Quantum representations of fundamental group of knot complement: a complete set of knot invariants?

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The idea about quantum representations of fundamental group of knot complement as a complete set of knot invariants was inspired by a popular article in New Scientist [4] summarizing briefly the recent situation in the field of knot invariants [5, 6, 7, 8, 9]. This field is of interest from the point of view of TGD because braids, links, and knots are closely related and because so called number theoretical braids have become the fundamental structure of quantum TGD [2]. One reason is that they define geometric correlates for the notion of finite measurement resolution. There are many other reasons [3]. Number theoretical braids are assigned to the incoming and outgoing lines of generalized Feynman diagrams so that topological QFT becomes part of quantum TGD.

The reading of the article inspired what looks like an attractive idea. Maybe a trivial one: any knot specialist could immediately tell this. The fundamental group of the complement of knot, call it G, is what is known as a complete knot invariant [5] in the sense that it fails to distinguish only between knot and its mirror image. The question is whether one could define a braided version of the fundamental group and perhaps define new Jones polynomial like knot invariants as quantum traces of the unitary quantum group representation matrices for G. If so, one could have a complete set of quantum invariants.

At least at first glimpse this seems be possible. One can assign to a braid a knot and more generally, links by joining the upper and lower ends of strands suitably. Knot is obtained by joining the upper end of n:th strand to lower end of n+1:th strand cyclically and assuming that the added connecting strands form a trivial braid having no braiding with the original braid gives a knot. The other extreme is N-link for N-braid obtained just by connecting the end points of strands. I do not remember how much non-uniqueness this representation involves. In any case this kind of representation [10] exists always for a given knot.

The following argument describes how the quantum representation for G could be constructed.

1. Let g be an element of G represented as a loop linked with the braid associated with the knot. Assume that the loop is un-knotted so that no new information having nothing to do with the knot (link) is brought in. Cut this loop in the region outside the braid from which knot (link) is obtained and join the ends to the upper and lower ends of the braid in such a manner that no new linking or knotting results. The new braid strand extends N-braid to N+1-braid. One can also connect the ends of the new strand by homotopically trivial un-knotted strand to get a closed loop. The outcome is 2-link in the case of knot and n+1-link in the case of n-link.

2. This assigns to the knot (link) and corresponding N-braid an N+1-braid and one can define braid/link/knot invariant as a quantum trace [8, 9]. Actually, any link invariant obtained as a

braid invariant defines a representation for a given element of G. In this manner all elements of G (there is infinite number of them) are represented as links and define new quantum invariants for the original knot. Since quantum phase is a root unity, many elements of G are expected to correspond to the same invariant.

3. The interpretation in terms of topological quantum field theory (TQFT) [11] suggests that one could add arbitrary numbers of loops representing elements of G so that infinite number of Jones invariants telling how these many particle states feel the presence of the braid defining the knot (or link). If all braids containing the braid defining the knot (link) as a sub-braid are allowed, a lot of irrelevant information is loaded to the system so that it seems natural to assume that the added loops are unknotted and mutually unlinked. The triviality of the braid associated with homotopy loops might imply that only single copy is needed for each element of G: the loops would effectively represent topological fermions. If also topological bosons are needed, super-symmetric arithmetic TQFT is what comes in mind first.

4. Large, perhaps even infinite number of braids would define braid/link/knot invariants for a given braid. This approach would conform with category theoretical thinking and would be in spirit with physicist's manner to get information about a physical system by perturbing it: all that we know from electrons is from its interactions with the rest of the word. In this case the perturbation would mean addition of loops representing elements of G. Knot would define the background in which the particles assigned with loops would "live". If mathematicians have not noticed this rather trivial construction of new knot and link invariants, the reason could be that usually the knot invariant is defined in terms of knot isolated from the rest of world.

References

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