Does the Modified Dirac Action Define the Fundamental Action Principle in TGD?

M. Pitkänen, August 14, 2004,
Dept. of Physics, University of Helsinki, Helsinki, Finland.
Email: matpitka@rock.helsinki.fi.
http://www.physics.helsinki.fi/∼matpitka/.

Contents

1 Introduction

In TGD [TGD, padTGD, cbookI, cbookII] Kähler function defines geometry of configuration space and thus also the quantum theory (see the first chapters of [TGD]). Although quantum criticality in principle predicts the possible values of Kähler coupling strength (analogous to critical temperature), one might hope that there exists even more fundamental approach involving no coupling constants and predicting even quantum criticality and realizing quantum gravitational holography. The Dirac determinant associated with the modified Dirac action is an excellent candidate in this respect [A1].

The modified Dirac operator indeed allows to realize quantum gravitational holography since it reduces to an effectively 3-dimensional Dirac operator by boundary conditions but depends on the normal derivatives of the imbedding space coordinates at the causal determinants, which are most naturally light like 3-surfaces.

The fact that the modes of the induced spinor fields have definite chiralities corresponding to conserved baryon and lepton number poses difficulties concerning the precise definition of the Dirac determinant and in quantum field theories one typically ends up with anomalies. It is possible to circumvent these difficulties in TGD framework and using the notion of bundle gerbe [1] one can develop a more cohomological argument demonstrating that these anomalies are absent in TGD.

The hermiticity of the modified Dirac operator at causal determinants requires that they are geodesic sub-manifolds. Hence there are good hopes of fixing the initial values of the normal derivatives of imbedding space coordinates at causal determinants uniquely and evaluating the exponent of Kähler function as a Dirac determinant without solving the field equations (quantum gravitational holography). If the spectrum of the modified Dirac operator is analogous to the bound state spectrum of hydrogen atom, there are good hopes that Dirac determinant is well-defined even without zeta function regularization. The normalization of the modified Dirac action by multiplying with a factor nR^4 with n a numerical factor could also bring in gravitational constant as $n \sim G^2/R^4$.

2 The modified Dirac equation

To begin with, recall that the field equations associated with Kähler action [A2], which is Maxwell action for the induced Kähler form having imbedding space coordinates as primary dynamical variables, read as

$$\begin{aligned} D_\beta(T^{\alpha\beta}h^k_\alpha) &- j^\alpha J^k_{\ l}\partial_\alpha h^l = 0 \ , \\ T^{\alpha\beta} &= J^{\nu\alpha}J_\nu^{\ \beta} - \frac{1}{4}g^{\alpha\beta}J^{\mu\nu}J_{\mu\nu} \ . \end{aligned} \quad (1)$$

The modified Dirac equation [A1] follows from the requirement of supersymmetry. Among other things this requires that the modified Dirac equation shares the enormous vacuum degeneracy of the Kähler action in the sense that equation becomes trivial for vacuum extremals. This requirement is satisfied if one replaces the induced gamma matrices in the Dirac action with modified ones so that one has

$$\begin{aligned} \hat{\Gamma}^\alpha D_\alpha\Psi &= 0 \ , \\ \hat{\Gamma}^\alpha &= T^\alpha_l\Gamma^l \ , \\ T^{\alpha k} &= \frac{\partial L_K}{\partial(\partial_\alpha h^k)} = T_K^{\alpha\beta}\partial_\beta h^k - J^{\alpha\beta}\partial_\alpha h^k - J^k_{\ l}\partial_\beta h^l g^{\alpha\beta} \ , \end{aligned} \quad (2)$$

where $T^{\alpha k}$ are generalized energy momentum currents associated with the Kähler action density L_K. $T^{\alpha k}$ decomposes into a part parallel to space-time surface and proportional to the Maxwell energy tensor and a part orthogonal to the space-time surface.

The modified Dirac equation reads as

$$\begin{aligned} \hat{\Gamma}^\alpha D_\alpha\Psi &= 0 \ , \\ \hat{\Gamma}^\alpha &= T^\alpha_{\ l}\Gamma^l \ . \end{aligned} \quad (3)$$

This equation follows from a modified Dirac action defined by the Lagrangian density

$$L = \overline{\Psi}\hat{\Gamma}^\alpha D_\alpha\Psi \ , \quad (4)$$

For this variational principle the induced gamma matrices are replaced with effective induced gamma matrices. Classical field equations imply that the modified gamma matrices are divergenceless:

$$D_\alpha \hat{\Gamma}^\alpha = 0 \tag{5}$$

and this guarantees hermiticity and supersymmetry. For the ordinary Dirac action this condition would lead to the minimal surface property. What sounds strange that the essentially hydrodynamical equations defined by Kähler action have fermionic counterpart: this is very far from intuitive expectations raised by the ordinary Dirac equation and something which one might not guess without taking super-symmetry very seriously.

The modified Dirac action cannot contain boundary term. This does not lead to a contradiction with the generation-genus correspondence since CP_2 type extremals providing a model of elementary particle are point like particles for all practical purposes. Hence it does not matter whether non-topological quantum numbers are at the boundaries or in the interior. Only the topology of the boundary component manifests itself in low energy physics by affecting the mass of the particle.

One can assign the modified Dirac action to any classical action and thus super-symmetrize it and construct also super-canonical charges. Only Kähler action is however consistent with the vanishing of the magnetic flux Hamiltonians for 3-surfaces whose CP_2 projections are Legendre sub-manifolds of CP_2 having vanishing induced Kähler form. Of course, vacuum functional defined as the exponent of Kähler function is not well-defined sense for all action principles: for instance, if the action is taken to be the 4-volume of the space-time surface, vacuum functional is well defined only if the four-volume is finite and this leads to a conflict with conservation laws. The vacuum degeneracy of the Kähler action is absolutely crucial for the topologization of the QFT Feynman rules by replacing the lines of Feynman diagrams with CP_2 type extremals so that on physical grounds the choice of the action principle is unique.

3 The exponent of Kähler function as Dirac determinant for the modified Dirac action?

There is quite fundamental and elegant interpretation of the modified Dirac action as a fundamental action principle. In this approach vacuum functional can be defined as the Grassmannian functional integral associated with the exponent of the modified Dirac action [A1]. This definition is invariant with respect to the scalings of the Dirac action so that theory contains no free parameters.

There are good reasons to expect that this functional integral is nothing but absolute minimum of the Kähler action with the value of Kähler coupling strength coming out as a prediction. Massless Dirac equation is satisfied in the sense that the vacuum expectation value of the modified Dirac action vanishes and implies Euler-Lagrange equations for Kähler action. The conservation of the canonical currents in turn implies the absolute minimization of Kähler action and c-number parts of the currents would correspond to the conserved currents associated with the Kähler action.

In second quantized formalism, which is more natural in TGD context, massless Dirac equation and Euler-Lagrange equations for the Kähler action follow from the variation of the modified Dirac action with respect to Ψ and imbedding space coordinates. This approach predicts non-vanishing conserved classical vacuum charges in accordance with the physical intuitions. The exponent of the Dirac action operating on vacuum state gives exponent of a term coming from the normal ordering expected to be identical with Kähler action.

The knowledge of the canonical currents and super-currents, together with the anti-commutation relations stating that the fermionic super-currents S_A and S_B associated with Hamiltonians H_A and H_B anti-commute to a bosonic current $H_{[A,B]}$, allows in principle to deduce the anti-commutation relations satisfied by the induced spinor field. In fact, these conditions replace the usual anti-commutation relations used to quantize free spinor field. Since the normal ordering of the Dirac action would give Kähler action,

Kähler coupling strength would be determined completely by the anti-commutation relations of the super-canonical algebra. Kähler coupling strength would be dynamical and the absolute minimization of Kähler action would be more or less equivalent with quantum criticality because criticality corresponds to conformal invariance and the quaternionic version of the super-conformal invariance results only for the extrema of Kähler action. p-Adic (or possibly more general) coupling constant evolution and quantum criticality would come out as a prediction whereas in the case that Kähler action is introduced as primary object, the value of Kähler coupling strength must be fixed by quantum criticality hypothesis.

The mixing of the M^4 chiralities of the imbedding space spinors serves as a signal for particle massivation and breaking of super-symmetry. The induced gamma matrices for the space-time surfaces which are deformations of M^4 indeed contain a small contribution from CP_2 gamma matrices: this implies a mixing of M^4 chiralities even for the modified Dirac action so that there is no need to introduce this mixing by hand.

4 Definition of the Dirac determinant

The integral over Grassmann variables gives a determinant of the modified Dirac operator which should reduce to the exponent of Kähler function in some sense perhaps involving TGD counterpart of regularization and renomarlization procedures. The attempt to define the Dirac determinant rigorously leads to unexpected insights giving real hopes that one can calculate the exponent of the Kähler function from local data.

4.1 The definition of Dirac determinant as a key to the understanding of the configuration space geometry

In TGD framework spinor modes have a definite H-chirality. As such the chiral symmetry is not a problem since a given matrix element of the modified Dirac operator is between spinor modes of the chirality and inner product is defined by the "time" component of conserved fermion current. The fermionic determinant by using zeta function regularization however requires the expression of the determinant as a product of the eigenvalues of the Dirac operator. The problem is that by the chirality condition the solution basis cannot correspond to the eigen modes of the modified Dirac operator in the ordinary sense ($D\Psi = \lambda\Psi$).

Usually this problem is avoided by defining the determinant as the square root of the determinant of the square of the Dirac operator. In TGD context the matrix defined by the square of the Dirac operator would transform quarks to leptons and this makes the trick not only ugly but intolerably un-physical.

TGD approach suggests and elegant solution of the problem. Consider first the constraints which the definition of the Dirac determinant should satisfy.

a) The definition should be general coordinate invariant. It should also realize quantum gravitational holography. Perhaps even in the sense that the value of the Kähler function should be in principle deducible from the data at the 3-dimensional causal determinants X^3 for which light-like 3-surfaces defining elementary particle horizons are excellent candidates. The hope is that the consistency conditions associated with definition of the Dirac determinant fix the normal derivatives $\partial_n h^k$ of the imbedding space-coordinates and hence the spectrum of the modified Dirac operator at X^3 and hence also the value of Kähler function.

b) The determinant should carry non-trivial information about the absolute minimum of Kähler action. Ordinary 3-dimensional Dirac operator for the induced spinors would not carry this information but the modified Dirac operator indeed does since the normal derivatives $\partial_n h^k$ of the imbedding space coordinates appear in the definition of energy momentum tensor appearing in the definition of the modified gamma matrices. Furthermore, the boundary condition states that the generalized energy momentum currents $T^{\alpha k}$ have a vanishing normal component at the causal determinant X^3:

$$T^{nk} = 0 \ . \tag{6}$$

This condition makes sense also for time-like boundaries and says that conserved charges must flow to the larger space-time sheets "at the end of the time".

Rather remarkably, by this boundary condition the modified Dirac equation involves only derivatives with respect to the coordinates x^i of X^3 and only the modified gamma matrices contain information about normal derivatives. Hence the idea of quantum gravitational holography suggests itself strongly.

c) The eigenvalues of the modified Dirac operator make sense if the exists a vector field n^α which is covariantly constant along X^3:

$$D_i n^\beta = 0 \ , \tag{7}$$

where α refers to the coordinates of X^3. These conditions might give the desired conditions fixing the values of the normal derivatives of the imbedding space coordinates and therefore also the value of the exponent of Kähler function as Dirac determinant. It is also natural to require that n^α is orthogonal to X^3 and thus defines "time" direction.

A natural identification for the vector field n^α is the gradient of the normal coordinate x^N constant at X^3: $n_\alpha = \delta_\alpha^N$. Covariant constancy gives conditions on Christoffel symbols at X^3:

$$\left\{ \begin{matrix} n \\ i \ j \end{matrix} \right\} = 0 \ . \tag{8}$$

The conditions state that $x^N = constant$ surface is not only a minimal 3-surface in X^4 but also a geodesic sub-manifold of X^4. These conditions are very strong and raise the hope that they could allow to realize quantum gravitational holography.

d) The eigenvalue equation reduces to the effectively 3-dimensional form at X^3

$$\begin{aligned} D\Psi \equiv \hat{\Gamma}^i D_i \Psi &= \lambda o \Psi \ , \\ o &= n^{\alpha}\Gamma_{\alpha} \ . \end{aligned} \tag{9}$$

Covariant constancy implies

$$[D, o] = 0 \ , \tag{10}$$

which implies Hermiticity and the possibility to orthonormalize the eigenmodes using the natural inner product. The orthonormalized inner product in turns allows to fix the anticommutators of the fermionic oscillator operators.

By operating to the modified Dirac equation by D one finds

$$D^2\Psi = \lambda^2 o^2 \Psi \ . \tag{11}$$

At time-like boundary the definition of the fermionic determinant would thus reduce to the standard one.

The construction makes sense also for the elementary particle horizons for which the normal defines light like vector so that the condition

$$o^2 = 0 \tag{12}$$

holds true and the spinor fields $Do\Psi$ define solutions of the modified Dirac equation $D\Psi = 0$ and there is a direct correspondence with the spectrum of Dirac operator and solutions of Dirac operator. This is undeniably something really nice.

A further elegant feature is that zero eigenvalues drop automatically from the determinant. This plus quantum gravitational holography makes the hypothesis that all causal determinants are actually light like 3-surfaces very attractive.

4.2 Is the regularization of Dirac determinant needed?

The definition of the Dirac determinant involves potential problems. In the case of ordinary Dirac operator the formal product of eigenvalues diverges and regularization is needed.

a) The first step consists of dividing the determinant with the determinant of the free Dirac operator D_0.

b) The next step is to express the resulting determinant D_A as an exponent of the trace of the logarithm of D_A, or effectively of the operator D_A/D_0 . If D is diagonalizable in some sense the trace reduces to the sum of the logarithms of the eigenvalues: $Tr(D) = \sum_n log(\lambda_n)$. This in turn can be expressed as a logarithmic derivative of the zeta function $\zeta(D) = \sum_n \lambda_n^{-s}$ defined by the eigenvalues of D at $s = 0$. This function can be defined by analytically continuing $\zeta(D)$ to the complex plane.

The modified Dirac operator is expected to deviate strongly from the ordinary Dirac operator and this might make its definition easier. For vacuum extremals all eigenvalues vanish and in this case Dirac determinant equals to unity even without regularization. This is consistent with the vanishing of the Kähler function.

One can imagine several manners of having a finite Dirac determinant.

1. The option based on hydrogen atom like bound state spectrum

If the allowed spinor modes are analogous to bound states localized inside light like boundary components, then even the possibility that the eigenvalues are bounded from above by a constant as in the case of hydrogen atom must be considered. In this case the asymptotic eigenvalue would be renormalizable to unity by multiplying the Dirac operator by a properly chosen constant. This could fix the value of the gravitational constant. Unfortunately, this option does not conform with the number theoretical vision.

2. The option based on a partially broken $\lambda \to \lambda$ symmetry

One might hope that time translations are replaced by scalings in TGD framework and the eigenvalues of the modified Dirac operator can be interpreted as scaling momenta. The symmetry $\lambda \leftrightarrow 1/\lambda$ would replace the $E \leftrightarrow -E$ symmetry of the ordinary Dirac operator transforming positive energy solutions to their negative energy counterparts. Exact symmetry would predict unit Dirac determinant and indeed a vanishing Kähler function. The non-vanishing value of Kähler function would correspond to a breaking of the conformal invariance so that $\lambda \leftrightarrow 1/\lambda$ symmetry would fail for a finite number of eigenvalues. Unfortunately, there is no real justification for the belief on partially broken $\lambda \leftrightarrow 1/\lambda$ symmetry.

3. The option inspired by the regularization of the ordinary Dirac determinant

The third and most realistic option is inspired by the regularization of the ordinary Dirac determinant by dividing it by the Dirac determinant of the free Dirac operator. Causal determinants involve always pairs of maximal strictly deterministic space-time regions, and the natural hypothesis is that the ratio of the Dirac determinants of the two adjacent deterministic space-time regions contains information about their Kähler actions.

The assumption the spectra of the modified Dirac operators of two regions coincide at the causal determinant apart from a finite number of eigenvalues whose ratio differs from unity is a strong statement about the character of the classical non-determinism. If the eigenvalue ratios are proportional to the difference of Kähler actions, Dirac determinant gives an exponent for the difference of Kähler actions of the two regions, and one can identify the result as the ratio of exponents of Kähler actions for the two regions. Nothing hinders from defining the regularized Dirac determinant for a given region as a corresponding finite exponent throwing away the common eigen values. This would give Dirac determinant as a global section of a determinant bundle.

This approach has several nice features.

a) The construction brings in mind the difference bundle construction giving rise to a non-trivial gerbe in turn defining a regularized Dirac determinant [1] as the multiplicative analog of gauge potential for which curvature form corresponds formation of the ratio of determinants and gives the ratios of determinants correctly. One could see this approach as the deeper one and Dirac determinants as a mere calculational trick to deduce the vacuum functional and Kähler action.

b) The value of the Dirac determinant does not depend on the normalization of the modified Dirac operator if the non-vanishing eigenvalues are in one-one correspondence.

c) There is also a consistency with the number theoretical conjectures about the relationship between Kähler coupling strength and gravitational constant discussed in the chapter "TGD as Generalized Number Theory" of [TGD].

4. Could G/R^2 be fixed from finiteness conditions?

The reason why options a) and b) looked so nice at first is following. Since the energy momentum tensor replaces metric tensor in the modification of the gamma matrices, contravariant modified gamma matrices have dimension $1/L^5$ instead of $1/L$ of the ordinary gamma matrices. Hence the modified Dirac action must be multiplied by a factor nR^4, n a numerical factor.

a) For option a) this constant should be such that the eigen values approach to unity. For option b) n could be determined by the requirement that almost exact conformal symmetry $\lambda \leftrightarrow 1/\lambda$ holds true, one can hope that n corresponds essentially to the ratio G^2/R^4, $\sqrt{G}/R \sim 10^{-4}$. In both cases the value of the gravitational constant would be fixed by the finiteness of the theory. In particular, the hypothesis is related to the number theoretical hypothesis relating Kähler coupling strength and gravitational constant to each other.

b) For option c) one could think (although reluctantly) of giving up the assumption about the one-one correspondence of eigen values the ratio of Dirac determinants so that some eigenvalues can become vanishing at the causal determinant. Hence the ratio of Dirac determinants would involve besides the product of ratios $\lambda_i^{1)}/\lambda_i^{2)}$ also "lonely" eigenvalues, say $\lambda_i^{1)}$ depending on the normalization of the modified Dirac operator. The ratio G/R^2 naturally appearing in the normalization could be fixed by the requirement that the "loners" are proportional to the exponent of Kähler action and for the deterministic regions with a vanishing Kähler action reduce to unity.

4.3 Could generalized index theorems provide information about the spectrum?

The presence of only a finite number of "active" eigen values for a given causal determinant enhances the hopes that information about the exponent of Kähler function could be deduced by using a generalization of index theorems [2]. Ordinary index theorems typically give the number of solutions $D\Psi = 0$ of the modified Dirac operator not expressible in the form $\Psi = D\Psi_1$ (covariantly constant right handed neutrino spinor with two spin directions is one example) in terms of topological invariants of the manifold.

For the option c) the conservative view would be that the generalized index theorem expresses the number of the eigenvalues which become vanishing in the transition between the adjacent regions. A more radical interpretation is that index theorem expresses the number of those solutions of the modified Dirac operators in the adjacent deterministic regions which correspond to different eigen values in terms of some natural topological invariant associated with the causal determinant.

The Chern-Simons action associated with the induced Kähler form defines a 3-connection form of 2-gerbe (see the chapter "Basic Extremals of the Kähler Action"). The more than obvious guess is that its value for a given causal determinant gives the number of "active" eigen values with sign telling the sign of asymmetry. Presumably the integer value of C-S action corresponds to a surface for which the projection of the causal determinant to CP_2 is many-to-one map.

The vanishing of the net C-S charge does not imply the vanishing of the Kähler function since only the value of the Kähler action for a particular maximal strictly non-deterministic region follows from the spectral asymmetry. If the value of the entire Kähler function were in question, Kähler function would be non-vanishing only when the dimension D of CP_2 projection would be $D = 4$ in some region in the interior of the space-time sheet.

If the CP_2 projection of the causal determinant is 2-dimensional, the spectral asymmetry is predicted to be trivial. The Kähler function could be non-vanishing for the space-time sheets having $D = 3$ everywhere. The regions of the space-time sheet with $D = 4$ (such as CP_2 extremals representing elementary particles and non-asymptotic regions of sheets having $D = 3$ asymptotically) would certainly contribute to the Kähler function.

4.4 About the conditions satisfied by the modified Dirac operators at the causal determinants

The discussion of the chapter "TGD as a Generalized Number Theory" inspires the idea that, at least for the causal determinants representing creation of matter from vacuum occurs, two space-time sheets branching from single 3-sheet are involved. This means that negative energy space-time sheets are time reflected as positive energy space-time sheets at the moment of "big bang". There are no initial nor final singularities nor not any big bang but just creation or annihilation of gravitational mass, and even living systems are excellent candidates sub-cosmologies consisting mainly of topological field quanta of classical fields.

Let the modified Dirac operator $D_\pm$ correspond to a space-time sheet possessing positive *resp.* negative time orientation. A similar association could hold true at the level of imbedding space for future *resp.* past directed light cones, which seem to play a key role in the construction of the configuration space geometry.

In accordance with option c), consider the ratio as the ratio of the determinants D_+ and D_-. The conditions making zeta function regularization un-necessary would be

$$\begin{aligned} [D_+, D_-] &= 0 \ , \\ (D_+ D_-^{-1})_{ij} &= \frac{\lambda_{+i}}{\lambda_{-i}} \delta i_{ij} \ . \end{aligned} \tag{13}$$

The condition implies the anti-commutator condition

$$\{D_+, D_-^{-1}\}_{ij} = 2\frac{\lambda_{+i}}{\lambda_{-i}} \delta i_{ij} \ . \tag{14}$$

D_+ (D_-^{-1}) is associated with the creation of positive (negative) energy space-time sheet would effectively act like a fermionic creation (annihilation operator). Also the interpretation as a super-symmetry might be considered.

Commutativity conditions allow also to consider the operator $D_+ D_-^{-1}$ as defining the "regularized" Dirac determinant giving the ratio of the exponents of Kähler action. Most eigenvalues of this operator would be equal to unity and the possible zero eigenvalues of D_+ would not contribute to the determinant.

Consider now the interpretation of these conditions for various kinds of causal determinants.

1. "Big bangs" and "big crushes" associated with sub-cosmologies

The first kind of causal determinants correspond to surfaces $\delta M^4_{\pm}(a) \times CP_2$ at which matter is created from vacuum. Here a labels the position of the light cone in M^4. The unions of these light cones form a category with the inclusion defining the arrow of geometric time and they define sectors $CH(\cup_i M^4_{\epsilon_i}(a_i))$, $\epsilon_i = \pm$, of the configuration space. In TGD based fractal cosmology $CH(M^4_{\epsilon_i}(a_i))$ defines a sub-cosmology in an extremely general sense.

A negative energy space-time sheet (D_-) would be time reflected as a positive energy space-time sheet (D_+). Besides $\lambda_+ = 1/\lambda_-$ satisfied apart from a finite number of eigenvalues, also the condition $o_+ o_- = 1$ for the contractions of normal vectors with induced gamma matrices would be satisfied. Lifted to vectors of the imbedding space the normal vectors are expected to be light like vectors with temporal components of opposite sign.

2. Elementary particle horizons and regions separating maximal strictly deterministic regions of space-time sheet

The wormhole contacts with Euclidian signature of metric are separated from the space-time sheets with a Minkowskian signature of induced metric by two elementary particle horizons with a distance of order of CP_2 length $R \sim 10^4$ Planck lengths. D_+ *resp.* D_- could be associated with the Minkowskian *resp.* Euclidian sides of the elementary particle horizon. For causal determinants separating maximal deterministic regions inside space-time sheet and having same signature of the induced metric an analogous treatment applies.

4.5 The global existence of Dirac determinant is not obvious

In gauge theories with chirality conservation there are also problems related to the global existence of the Dirac determinant as a functional of background gauge field A when one induces the determinant bundle to the space of gauge equivalence classes of gauge potentials. Since chirality condition is involved also now and since the signs of eigenvalues can be also negative, this problem might be present also now.

In the recent case the space of connections divided by gauge group is replaced with the configuration space, and the task is to demonstrate that the fermionic determinant expected to reduce to an exponent of the Kähler function is indeed single-valued. The gauge theory situation is discussed in excellent manner in [1] and the following discussion represents basic ideas plus a straightforward translation to recent case without any attempt to really prove anything.

4.5.1 Spectral flow through origin as the basic problem

The difficulties are basically due to the fact that the spectrum of the Dirac operator contains both negative and positive part, and it can happen that some eigenvalues change sign as a functional of vector potential. This is not as such a catastrophe since the resulting Fock spaces are canonically isomorphic but there can be obstructions for the global existence of the Dirac determinant.

The naive expectation is that the exponent of Kähler function changes sign when an eigenvalue changes sign. This is however not possible if Kähler function is real. Hence there is a temptation to conclude that the determinant is trivially globally defined for given values of zero modes. Also the absolute minimization of Kähler action suggests that the determinant is negative for all 3-surfaces (or rather space-time sheets) and cannot change sign. Regularization by analytic continuation might spoil this argument.

4.5.2 Global existence of the Dirac determinant from triviality of bundle gerbe

The conditions for the global existence of the Dirac determinant can be formulated as a condition that the so called bundle gerbe [1]. Gerbe is a generalization of connection 1-form to a connection 2-form, and would now be defined in the configuration space. The global existence of the Dirac determinant states that that gerge is trivial (”pure gauge”), that is expressible as an exterior derivative of 1-form [1]. Gerbe is non-trivial only if the third cohomology group $H^3(CH)$ of the configuration space is trivial just as the non-triviality of second cohomology group is necessary for the existence of magnetic monopole. This is the case if the configuration space is Kähler manifold.

4.5.3 About rigorous definition of Dirac determinant

The rigorous definition of the Dirac determinant [1] forces to consider the subspaces $H_+(X^3, \lambda)$ consisting of eigen modes of Dirac operator with eigenvalues of Dirac operator strictly larger than λ, as a functional of 3-surface having fixed values of zero modes.

Second key notion is the Grassmann manifold $Gr_p(H_+)$ consisting of closed sub-spaces $W \subset H$ for which $P_W - P_{H_+}$ is in the Schatten ideal L_p of operators T with $|T|^p$ a trace class operator. The spaces $H_+(\lambda)$ belong to $Gr_p(H_+)$ for $p > dim(M)$, where M is the dimension of the manifold in which Dirac operator is defined.

By quantum-gravitational holography one has $dim(M) = 3$ now: this is essentially due to the special properties of the modified Dirac operator. What makes this so interesting is that the type II_1 factors of von Neumann algebras which appear in the construction of quantum TGD, have in a well defined sense a discrete valued fractal dimension $D < 4$. Gr_p allows an extension to a complex line bundle defined by single element of second cohomology group which can be regarded as Dirac determinant.

One can cover X^3 with the open sets $U_\lambda = \{X^3 | \lambda \notin Spec(D(X^3))\}$ (λ does not belong to the spectrum of D). In each U_λ the map $X^3 \rightarrow H_+(X^3, \lambda)$ allows to define a determinant bundle in fixed zero modes sector of the configuration space via pull back.

Furthermore, one can define difference bundles in the intersections $U_{\lambda\lambda'} = U_\lambda \cap U_{\lambda'}$ and the determinant in the intersection is essentially the ratio $DET(X^3, \lambda)/DET(X^3, \lambda')$ and thus the product of eigenvalues in the open interval (λ, λ') and well-defined even without any regularization.

This system of difference bundles satisfies in the triple intersections

$$U_{\lambda\lambda'\lambda''} = U_\lambda \cap U_{\lambda'} U_{\lambda''}$$

the cocycle conditions

$$DET_{p,\lambda\lambda'} DET_{p,\lambda'\lambda''} = DET_{p,\lambda\lambda''} \quad (15)$$

guaranteing that the differences are single valued.

This cocycle property defines what is known as a trivial gerbe. The triviality follows from the decomposition to the difference of bundles and corresponds to the ability to express connection 2-form in the configuration space in terms of 1-form. A good guess is that this connection two-form is just the Kähler form of configuration space. The triviality of course requires the existence of $DET_{p\lambda}$ and the this might require a regularization procedure.

The Fock spaces $\mathcal{F}(X^3, \lambda)$ are canonically isomorphic for different values of λ. Hence the change of the sign of eigenvalue is as such not a catastrophe since one can map the fibers to each other. Using physics terminology, the Dirac sea filled up to λ is equivalent with the Dirac sea filled up to λ'. The isomorphism mediating this filling brings in the Dirac determinant a determinant R of a unitary transformation mixing the eigen modes with eigenvalues in the range (λ, λ'). For a trivial bundle gerbe the emergence of this determinant does not lead to non-uniqueness.

References

[TGD] M. Pitkänen (1995) *Topological Geometrodynamics* Internal Report HU-TFT-IR-95-4 (Helsinki University). http://www.physics.helsinki.fi/∼matpitka/.

[padTGD] M. Pitkänen (1995), *Topological Geometrodynamics and p-Adic Numbers*. Internal Report HU-TFT-IR-95-5 (Helsinki University). http://www.physics.helsinki.fi/∼matpitka/padtgd.html.

[cbookI] M. Pitkänen (1998), *TGD inspired theory of consciousness with applications to bio-systems*. http://www.physics.helsinki.fi/∼matpitka/cbook.html .

[cbookII] M. Pitkänen (2001), *Genes, Memes, Qualia, and Semitrance*. http://www.physics.helsinki.fi/∼matpitka/cbookII.html .

[1] J. Mickelson (2002), *Gerbes, (Twisted) K-Theory, and the Supersymmetric WZW Model*, hep-th/0206139.

[2] Atyiah-Singer index-theorem, http://en.wikipedia.org/wiki/Atiyah-Singer_index_theorem.

[A1] Chapter "Configuration Space Spinor Structure" of [TGD].

[A2] Chapter "Basic Extremals of the Kähler Action" of [TGD].

[A3] Chapter "TGD as Generalized Number Theory" of [TGD].