Langlands conjectures in TGD framework

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Contents

Abstract

The arguments of this article support the view that in TGD Universe number theoretic and geometric Langlands conjectures could be understood very naturally. The basic notions are following.

1. Zero energy ontology (ZEO) and the related notion of causal diamond CD (CD is short hand for the cartesian product of causal diamond of M^4 and of CP_2). ZEO leads to the notion of partonic 2-surfaces at the light-like boundaries of CD and to the notion of string world sheet. These notions are central in the recent view about TGD. One can assign to the partonic 2-surfaces a conformal moduli space having as additional coordinates the positions of braid strand ends (punctures). By electric-magnetic duality this moduli space must correspond closely to the moduli space of string world sheets.

2. Electric-magnetic duality realized in terms of string world sheets and partonic 2-surfaces. The group G and its Langlands dual ${}^LG$ would correspond to the time-like and space-like braidings. Duality predicts that the moduli space of string world sheets is very closely related to that for the partonic 2-surfaces. The strong form of 4-D general coordinate invariance implying electric-magnetic duality and S-duality as well as strong form of holography indeed predicts that the collection of string world sheets is fixed once the collection of partonic 2-surfaces at light-like boundaries of CD and its sub-CDs is known.

3. The proposal is that finite measurement resolution is realized in terms of inclusions of hyperfinite factors of type II_1 at quantum level and represented in terms of confining effective gauge group [14]. This effective gauge group could be some associate of G: gauge group, Kac-Moody group or its quantum counterpart, or so called twisted quantum Yangian strongly suggested by twistor considerations. At space-time level the finite measurement resolution would be represented in terms of braids at space-time level which come in two varieties correspond to braids assignable to space-like surfaces at the two light-like boundaries of CD and with light-like 3-surfaces at which the signature of the induced metric changes and which are identified as orbits of partonic 2-surfaces connecting the future and past boundaries of CDs.

   There are several steps leading from G to its twisted quantum Yangian. The first step replaces point like particles with partonic 2-surfaces: this brings in Kac-Moody character. The second step brings in finite measurement resolution meaning that Kac-Moody type algebra is replaced with its quantum version. The third step brings in zero energy ontology: one cannot treat single partonic surface or string world sheet as independent unit: always the collection of partonic 2-surfaces and corresponding string worlds sheets defines the geometric structure so that multilocality and therefore quantum Yangian algebra with multilocal generators is unavoidable.

   In finite measurement resolution geometric Langlands duality and number theoretic Langlands duality are very closely related since partonic 2-surface is effectively replaced with the punctures representing the ends of braid strands and the orbit of this set under a discrete subgroup of G defines effectively a collection of "rational" 2-surfaces. The number of the "rational" surfaces in geometric Langlands conjecture replaces the number of rational points of partonic 2-surface in its number theoretic variant. The ability to compute both these numbers is very relevant for quantum TGD.

4. The natural identification of the associate of G is as quantum Yangian of Kac-Moody type group associated with Minkowskian open string model assignable to string world sheet representing a string moving in the moduli space of partonic 2-surface. The dual group corresponds to Euclidian string model with partonic 2-surface representing string orbit in the moduli space of the string world sheets. The Kac-Moody algebra assigned with simply laced G is obtained using the standard tachyonic free field representation obtained as ordered exponentials of Cartan algebra generators identified as transversal parts of M^4 coordinates for the braid strands. The importance of the free field representation generalizing to the case of non-simply laced groups in the realization of finite measurement resolution in terms of Kac-Moody algebra cannot be over-emphasized.

5. Langlands duality involves besides harmonic analysis side also the number theoretic side. Galois groups (collections of them) defined by infinite primes and integers having representation as symplectic flows defining braidings. I have earlier proposed that the hierarchy of these Galois groups define what might be regarded as a non-commutative homology and cohomology. Also G has this kind of representation which explains why the representations of these two kinds of groups are so intimately related. This relationship could be seen as a generalization of the MacKay correspondence between finite subgroups of $SU(2)$ and simply laced Lie groups.

6. Symplectic group of the light-cone boundary acting as isometries of the WCW geometry [3] allowing to represent projectively both Galois groups and symmetry groups as symplectic flows so that the non-commutative cohomology would have braided representation. This leads to braided counterparts for both Galois group and effective symmetry group.

7. The moduli space for Higgs bundle playing central role in the approach of Witten and Kapustin to geometric Landlands program is in TGD framework replaced with the conformal moduli space for partonic 2-surfaces. It is not however possible to speak about Higgs field although moduli defined the analog of Higgs vacuum expectation value. Note that in TGD Universe the most natural assumption is that all Higgs like states are "eaten" by gauge bosons so that also photon and gluons become massive. This mechanism would be very general and mean that massless representations of Poincare group organize to massive ones via the formation of bound states. It might be however possible to see the contribution of p-adic thermodynamics depending on genus as analogous to Higgs contribution since the conformal moduli are analogous to vacuum expectation of Higgs field.

1 Introduction

Langlands program [3, 10, 9] relies on very general conjectures about a connection between number theory and harmonic analysis relating the representations of Galois groups with the representations of certain kinds of Lie groups to each other. Langlands conjecture has many forms and it is indeed a conjecture and many of them are imprecise since the notions involved are not sharply defined.

Peter Woit noticed that Edward Frenkel had given a talk with rather interesting title "What do Fermat's Last Theorem and Electro-magnetic Duality Have in Common?" [6]? I listened the talk and found it very inspiring. The talk provides bird's eye of view about some basic aspects of Langlands program using the language understood by physicist. Also the ideas concerning the connection between Langlands duality and electric-magnetic duality generalized to S-duality in the context of non-Abelian gauge theories and string theory context and developed by Witten and Kapustin [14] and followers are summarized. In this context $D = 4$ and twisted version of $\mathcal{N} = 4$ SYM familiar from twistor program and defining a topological QFT appears.

For some years ago I made my first attempt to understand what Langlands program is about and tried to relate it to TGD framework [9]. At that time I did not really understand the motivations for many of the mathematical structures introduced. In particular, I did not really understand the motivations for introducing the gigantic Galois group of algebraic numbers regarded as algebraic extension of rationals.

1. Why not restrict the consideration to finite Galois groups [2] or their braided counterparts (as I indeed effectively did [9])? At that time I concentrated on the question what enormous Galois group of algebraic numbers regarded as algebraic extension of rationals could mean, and proposed that it could be identified as a symmetric group consisting of permutations of infinitely many objects. The definition of this group is however far from trivial. Should one allow as generators of the group only the permutations affecting only finite number of objects or permutations of even infinite number of objects?

   The analogous situation for the sequences of binary digits would lead to a countable set of sequence of binary digits forming a discrete set of finite integers in real sense or to 2-adic integers forming a 2-adic continuum. Something similar could be expected now. The physical constraints coming the condition that the elements of symmetric group allow lifting to braidings suggested that the permutations permuting infinitely many objects should be periodic meaning that the infinite braid decomposes to an infinite number of identical N-braids and braiding is same for all of them. The p-adic analog would be p-adic integers, which correspond to rationals having periodical expansion in powers of p. Braids would be therefore like pinary digits. I regarded this choice as the most realistic one at that time. I failed to realize the possibility of having analogs of p-adic integers by general permutations. In any case, this observation makes clear that the unrestricted Galois group is analogous to a Lie group in topology analogous to p-adic topology rather than to discrete group. Neither did I realize that the Galois groups could be finite and be associated with some other field than rationals, say a Galois group associated with the field of polynomials of n-variable with rational coefficients and with its completion with coefficients replaced by algebraic numbers.

2. The ring of adeles [1] can be seen as a Cartesian product of non-vanishing real numbers $R_\times$ with the infinite Cartesian product $\prod Z_p$ having as factors p-adic integers Z_p for all values of prime p. Rational adeles are obtained by replacing R with rationals Q and requiring that multiplication of rational by integers is equivalent with multiplication of any Z_p with rational. Finite number of factors in Z_p can correspond to Q_p: this is required in to have finite adelic norm defined as the product of p-adic norms. This definition implicitly regards rationals as common to all number fields involved. At the first encounter with adeles I did not realize that this definition is in spirit with the basic vision of TGD.

   The motivation for the introduction of adele is that one can elegantly combine the algebraic groups assignable to rationals (or their extensions) and all p-adic number fields or even more general function fields such as polynomials with some number of argument at the same time as a Cartesian product of these groups as well as to finite fields. This is indeed needed if one wants to realize number theoretic universality which is basic vision behind physics as generalized number theory vision. This approach obviously means enormous economy of thought irrespective of whether one takes adeles seriously as a physicist.

In the following I will discuss Taniyama-Shimura-Weil theorem and Langlands program from TGD point view.

2 Taniyama-Shimura-Weil conjecture from the perspective of TGD

2.1 Taniyama-Shimura-Weil theorem

It is good to consider first the Taniyama-Shimura-Weil conjecture [5] from the perspective provided by TGD since this shows that number theoretic Langlands conjecture could be extremely useful for practical calculations in TGD framework.

1. Number theoretical universality requires that physics in real number field and various p-adic number fields should be unified to a coherent hole by a generalization of the notion of number: different number fields would be like pages of book intersecting along common rationals. This would hold true also for space-time surfaces and imbedding space but would require some preferred coordinates for which rational points would determined the intersection of real and p-adic worlds. There are good reasons for the hypothesis that life resides in the intersection of real and p-adic worlds.

   The intersection would correspond at the level of partonic 2-surfaces rational points of these surfaces in some preferred coordinates, for which a finite-dimensional family can be identified on basis of the fundamental symmetries of the theory. Allowing algebraic extensions one can also consider also some algebraic as common points. In any case the first question is to count the number of rational points for a partonic 2-surface.

   2-dimensional Riemann surfaces serve also as a starting point of number theoretic Langlands problem and the same is true for the geometric Langlands program concentrating on Riemann surfaces and function fields defined by holomorphic functions.

2. The number theoretic side of Taniyama-Shimura-Weil (TSW briefly) theorem for elliptic surfaces, which is essential for the proof of Fermat's last theorem, is about counting the integer (or equivalently rational) points of the elliptic surfaces

   $$y^2 = x^3 + ax + b \ \ , \ \ a, b \in Z \ \ .$$

   The theorem relates number theoretical problem to a problem of harmonic analysis, which is about group representations. What one does is to consider the above Diophantine equation modulo p for all primes p. Any solution with finite integers smaller than p defines a solution in real sense if $mod\ \ p$ operation does not affect the equations. Therefore the existence of a finite number of solutions involving finite integers in real sense means that for large enough p the number a_p of solutions becomes constant.

3. On harmonic analysis one studies so called modular forms $f(\tau)$, where τ is a complex coordinate for upper half plane defining moduli space for the conformal structures on torus. Modular forms have well defined transformation properties under group $Gl_2(R)$: the action is defined by the formula $\tau \rightarrow (a\tau+b)/(c\tau+d)$. The action of $Gl_2(Z)$ or its appropriate subgroup is such that the modular form experiences a mere multiplication by a phase factor: $D(hk) = c(h,k)D(h)D(k)$. The phase factors obey cocycle conditions $D(h,k)D(g,hk) = D(gh,k)D(g,h)$ guaranteeing the associativity of the projective representation.

   Modular transformations are clearly symmetries represented projectively as quantum theory indeed allows to do. The geometric interpretation is that one has projective reprseentations in the fundamental domain of upper plane defined by the identification of the points differing by modular transformations. In conformally symmetric theories this symmetry is essential. Fundamental domain is analogous to lattice cell. One often speaks of cusp forms: cusp forms vanish at the boundary of the fundamental domain defined as the quotient of the upper half plane by a subgroup -call it Γ of the modular group $Sl_2(Z)$. The boundary corresponds to $Im(\tau) \rightarrow \infty$ or equivalently $q = exp(i2\pi\tau) \rightarrow 0$.

Remark: In TGD framework modular symmetry says that elementary particle vacuum functionals are modular invariants. For torus one has the above symmetry but for Riemann surface with higher genus modular symmetries correspond to a subgroup of $Sl_{2g}(Z)$.

4. One can expand the modular form as Fourier expansion using the variable $q = exp(i2\pi\tau)$ as

$$f(\tau) = \sum_{n>0} b_n q^n \ .$$

$b_1 = 1$ fixes the normalization. $n > 0$ in the sum means that the form vanishes at the boundary of the fundamental domain associated with the group Γ. The TSW theorem says that for prime values $n = p$ one has $b_p = a_p$, where a_p is the number of mod p integer solutions to the equations defining the elliptic curve. At the limit $p \rightarrow \infty$ one obtains the number of real actual rational points of the curve if this number is finite. This number can be also infinite. The other coefficients b_n can be deduced from their values for primes since b_n defines what is known as a multiplicative character in the ring of integers implying $b_{mn} = b_m b_n$ meaning that b_n obeys a decomposition analogous to the decomposition of integer into a product of primes.

The definition of the multiplicative character is extremely general: for instance it is possible to define quantum counterparts of multiplicative characters and of various modular forms by replacing integers with quantum integers defined as products of quantum primes for all primes except one -call it p_0, which is replaced with its inverse: this definition of quantum integer appears in the deformation of distributions of integer valued random variable characterized by rational valued parameters and is motivated by strange findings of Shnoll [1]. The interpretation could be in terms of TGD based view about finite measurement resolution bringing in quantum groups and also preferred p-adic prime naturally.

5. TSW theorem allows to prove Fermat's last theorem: if the latter theorem were wrong also TSW theorem would be wrong. What also makes TSW theorem so wonderful is that it would allow to count the number of rational points of elliptic surfaces just by looking the properties of the automorphic forms in $Gl_2(R)$ or more general group. A horrible looking problem of number theory is transformed to a problem of complex analysis which can be handled by using the magic power of symmetry arguments. This kind of virtue does not matter much in standard physics but in quantum TGD relying heavily on number theoretic universality situation is totally different. If TGD is applied some day the counting of rational points of partonic surfaces is everyday practice of theoretician.

2.2 How to generalize TSW conjecture?

The physical picture of TGD encourages to imagine a generalization of the Tanyama-Shimura-Weil conjecture.

1. The natural expectation is that the conjecture should make sense for Riemann surfaces of arbitrary genus g instead of $g = 1$ only (elliptic surfaces are tori). This suggests that one should one replace the upper half plane representing the moduli space of conformal equivalence classes of toric geometries with the $2g$-dimensional (in the real sense) moduli space of genus g conformal geometries identifiable as Teichmüller space.

   This moduli space has symplectic structure analogous to that of $g + g$-dimensional phase space and this structure relates closely to the cohomology defined in terms of integrals of holomorphic forms over the $g + g$ cycles which each handle carrying two cycles. The moduli are defined by the values of the holomorphic one-forms over the cycles and define a symmetric matrix Ω_{ij} (modular parameters), which is modular invariant [4]. The modular parameters related $Sp_2g(Z)$ transformation correspond to same conformal equivalence class.

   If Galois group and effective symmetry group G are representable as symplectic flows at the light-like boundary of $CD(\times CP_2)$, their action automatically defines an action in the moduli space. The action can be realized also as a symplectic flow defining a braiding for space-like braids assignable to the ends of the space-time surface at boundaries of CD or for time-like braids assignable to light-like 3-surfaces at which the signature of the induced metric changes and identified as orbits of partonic 2-surfaces analogous to black hole horizons.

2. It is possible to define modular forms also in this case. Most naturally they correspond to theta functions used in the construction of elementary particle functionals in this space [4]. Siegel modular forms transform naturally under the symplectic group $Sp_{2g}(R)$ and are projectively invariant $Sp_{2g}(Z)$. More general moduli spaces are obtained by allowing also punctures having interpretation as the ends of braid strands and very naturally identified as the rational points of the partonic 2-surface. The modular forms defined in this extended moduli space could carry also information about the number of rational points in the same manner as the automorphic representations of $Gl_2(R)$ carry information about the number of rational points of elliptic curves.

3. How Tanyama-Shimura-Weil conjecture should be generalized? Also now one can consider power series of modular forms with coefficients b_n defining multiplicative characters for the integers of field in question. Also now the coefficients a_p could give the number of integer/rational points of the partonic 2-surface in mod p approximation and at the limit $p \rightarrow \infty$ the number of points a_p would approach to a constant if the number of points is finite.

4. The only sensical interpretation is that the analogs of elementary particle vacuum functionals [4] identified as modular forms must be always restricted to partonic 2-surfaces having the same number of marked points identifiable as the end points of braid strands rational points. It also seems necessary to assume that the modular forms factorize to a products of two parts depending on Teichmüller parameters and positions of punctures. The assignment of fermionic and bosonic quantum numbers with these points conforms with this interpretation. As a special case these points would be rational. The surface with given number or marked points would have varying moduli defined by the conformal moduli plus the positions of the marked points. This kind of restriction would be physically very natural since it would mean that only braids with a given number of braid strands ending at fixed number of marked points at partonic 2-surfaces are considered in given quantum state. Of course, superpositions of these basis states with varying braid number would be allowed.

3 Unified treatment of number theoretic and geometric Langlands conjectures in TGD framework

One can already now wonder what the relationship of the TGD view about number theoretic Langlands conjecture to the geometric Langlands conjecture could be?

1. Finite measurement resolution conjectured to be definable in terms of effective symmetry group G defined by the inclusion of hyper-finite factors of type II_1 [14](HFFs in the sequel) effectively replaces partonic 2-surfaces with collections of braid ends and the natural idea is that the orbits of these collections under finite algebraic subgroup of symmetry group defining finite measurement resolution gives rise to orbit with finite number of points (point understood now as collection of rational points). The TGD variant of the geometric Langlands conjecture would allow to deduce the number of different collections of rational braid ends for the quantum state considered (one particular WCW spinor field) from the properties of automorphic form.

2. Quantum group structure is associated with the inclusions of HFFs, with braid group representations, integrable QFTs, and also with the quantum Yangian symmetry [15, 13] suggested strongly by twistor approach to TGD. In zero energy ontology physical states define Lie-algebra and the multil-ocality of the scattering amplitudes with respect to the partonic 2-surfaces (that is at level of WCW) suggests also quantum Yangian symmetry. Therefore the Yangian of the Kac-Moody type algebra defining measurement resolution is a natural candidate for the symmetry considered. What is important is that the group structure is associated with a finite-dimensional Lie group.

This picture motivates the question whether number theoretic and geometric Langlands conjecture could be realized in the same framework? Could electric-magnetic duality generalized to S-duality bring in the TGD counterpart of effective symmetry group G in some manner. This framework would be considerably more general than the 4-D QFT framework suggested by Witten and Kapustin [14] and having very close analogies with TGD view about space-time.

The following arguments support the view that in TGD Universe number theoretic and geometric Langlands conjectures could be understood very naturally. The basic notions are following.

1. Zero energy ontology and the related notion of causal diamond CD (CD is short hand for the cartesian product of causal diamond of M^4 and of CP_2). This notion leads to the notion of partonic 2-surfaces at the light-like boundaries of CD and to the notion of string world sheet.

2. Electric-magnetic duality realized in terms of string world sheets and partonic 2-surfaces. The group G and its Langlands dual ${}^LG$ would correspond to the time-like and space-like braidings. Duality predicts that the moduli space of string world sheets is very closely related to that for the partonic 2-surfaces. The strong form of 4-D general coordinate invariance implying electric-magnetic duality and S-duality as well as strong form of holography indeed predicts that the collection of string world sheets is fixed once the collection of partonic 2-surfaces at light-like boundaries of CD and its sub-CDs is known.

3. The proposal is that finite measurement resolution is realized in terms of inclusions of hyperfinite factors of type II_1 at quantum level and represented in terms of confining effective gauge group [14]. This effective gauge group could be some associate of G: gauge group, Kac-Moody group or its quantum counterpart, or so called twisted quantum Yangian strongly suggested by twistor considerations ("symmetry group" hitherto). At space-time level the finite measurement resolution would be represented in terms of braids at space-time level which come in two varieties correspond to braids assignable to space-like surfaces at the two light-like boundaries of CD and with light-like 3-surfaces at which the signature of the induced metric changes and which are identified as orbits of partonic 2-surfaces connecting the future and past boundaries of CDs.

   There are several steps leading from G to its twisted quantum Yangian. The first step replaces point like particles with partonic 2-surfaces: this brings in Kac-Moody character. The second step brings in finite measurement resolution meaning that Kac-Moody type algebra is replaced with its quantum version. The third step brings in zero energy ontology: one cannot treat single partonic surface or string world sheet as independent unit: always the collection of partonic 2-surfaces and corresponding string worlds sheets defines the geometric structure so that multilocality and therefore quantum Yangian algebra with multilocal generators is unavoidable.

   In finite measurement resolution geometric Langlands duality and number theoretic Langlands duality are very closely related since partonic 2-surface is effectively replaced with the punctures representing the ends of braid strands and the orbit of this set under a discrete subgroup of G defines effectively a collection of "rational" 2-surfaces. The number of the "rational" surfaces in geometric Langlands conjecture replaces the number of rational points of partonic 2-surface in its number theoretic variant. The ability to compute both these numbers is very relevant for quantum TGD.

4. The natural identification of the associate of G is as quantum Yangian of Kac-Moody type group associated with Minkowskian open string model assignable to string world sheet representing a string moving in the moduli space of partonic 2-surface. The dual group corresponds to Euclidian string model with partonic 2-surface representing string orbit in the moduli space of the string world sheets. The Kac-Moody algebra assigned with simply laced G is obtained using the standard tachyonic free field representation obtained as ordered exponentials of Cartan algebra generators identified as transversal parts of M^4 coordinates for the braid strands. The importance of the free field representation generalizing to the case of non-simply laced groups in the realization of finite measurement resolution in terms of Kac-Moody algebra cannot be over-emphasized.

5. Langlands duality involves besides harmonic analysis side also the number theoretic side. Galois groups (collections of them) defined by infinite primes and integers having representation as symplectic flows defining braidings. I have earlier proposed that the hierarchy of these Galois

groups define what might be regarded as a non-commutative homology and cohomology. Also G has this kind of representation which explains why the representations of these two kinds of groups are so intimately related. This relationship could be seen as a generalization of the MacKay correspondence between finite subgroups of $SU(2)$ and simply laced Lie groups.

6. Symplectic group of the light-cone boundary acting as isometries of the WCW geometry [3] allowing to represent projectively both Galois groups and symmetry groups as symplectic flows so that the non-commutative cohomology would have braided representation. This leads to braided counterparts for both Galois group and effective symmetry group.

7. The moduli space for Higgs bundle playing central role in the approach of Witten and Kapustin to geometric Landlands program [14] is in TGD framework replaced with the conformal moduli space for partonic 2-surfaces. It is not however possible to speak about Higgs field although moduli defined the analog of Higgs vacuum expectation value. Note that in TGD Universe the most natural assumption is that all Higgs like states are "eaten" by gauge bosons so that also photon and gluons become massive. This mechanism would be very general and mean that massless representations of Poincare group organize to massive ones via the formation of bound states. It might be however possible to see the contribution of p-adic thermodynamics depending on genus as analogous to Higgs contribution since the conformal moduli are analogous to vacuum expectation of Higgs field.

3.1 Number theoretic Langlands conjecture in TGD framework

Number theoretic Langlands conjecture generalizes TSW conjecture to a duality between two kinds of groups.

1. At the number theoretic side of the duality one has an n-dimensional representation of Galois group for the algebraic numbers regarded as algebraic extension of rationals. In the more general case one can consider arbitrary number field identified as algebraic extension of rationals. One can assign to the number field its rational adele. In the case of rationals this brings in both real numbers and p-adic numbers so that huge amount of information can be packed to the formulas. For anyone who has not really worked concretely with number theory it is difficult to get grasp of the enormous generality of the resulting theory.

2. At the harmonic analysis side of the conjecture one has n-dimensional representation of possibly non-compact Lie group G and its Langlands dual ${}^LG$ appearing also in the non-Abelian form of electric-magnetic duality. The idea that electric-magnetic duality generalized to S-duality could provide a physical interpretation of Langlands duality is suggestive. $U(n)$ is self dual in Langlands sense but already for $G = SU(3)$ one has ${}^LG = SU(3)/Z_3$. For most Lie groups the Lie algebras of G and ${}^LG$ are identical but even the Lie algebras can be different. $Gl_2(R)$ is replaced with any reductive algebraic group and in the matrix representation of the group the elements of the group are replaced by adeles of the discrete number field considered.

3. Langlands duality relates the representations of the Galois group in question to the automorphic representations of G. The action of the Lie group is on the argument of the modular form so that one obtains infinite-dimensional representation of G for non-compact G analogous to a unitary representation of Lorentz group. The automorphic forms are eigenstates of the Casimir operator of G. Automorphy means that a subgroup Γ of the modular group leaves the automorphic form invariant modulo phase factor.

4. The action of the modular transformation $\tau \rightarrow -1/\tau$ in the case of $Gl_2(R)$ replaces G with ${}^LG$. In the more general case (for the moduli space of Riemann surfaces of genus g possessing n punctures) the definition of the modular transformation induce the change $G \rightarrow^L G$ does not look obvious. Even the idea that one has only two groups related by modular transformation is not obvious. For electromagnetic duality with τ interpreted in terms of complexified gauge coupling strength this interpretational problem is not encountered.

3.2 Geometric Langlands conjecture in TGD framework

Consider next the geometric Langlands conjecture from TGD view point.

1. The geometric variant of Langlands conjecture replaces the discrete number field F (rationals and their algebraic extensions say) with function number field- say rational function with rational coefficients- for which algebraic completion defines the gigantic Galois group. Witten and Kapustin [14] proposed a concrete vision about how electric-magnetic duality generalized to S-duality could allow to understand geometric Langlands conjecture.

2. By strong form of general coordinate invariance implying holography the partonic 2-surfaces and their 4-D tangent space data (not completely free probably) define the basic objects so that WCW reduces to that for partonic 2-surfaces so that the formulation of geometric Langlands conjecture for the local field defined by holomorphic rational functions with rational coefficients at partonic 2-surface might make sense.

3. What geometric Langlands conjecture could mean in TGD framework? The transition from space-time level to the level of world of classical worlds suggests that polynomials with rational functions with rational coefficients define the analog of rational numbers which can be regarded to be in the intersection real and p-adic WCWs. Instead of counting rational points of partonic 2-surface one might think of counting the numbers of points in the intersection of real and p-adic WCWs in which life is suggested to reside. One might well consider the possibility that a kind of volume like measure for the number of these point is needed. Therefore the conjecture would be of extreme importance in quantum TGD. Especially so if the intersection of real and p-adic worlds is dense subset of WCW just as rationals form a dense subset of reals and p-adic numbers.

3.3 Electric-magnetic duality in TGD framework

Consider first the ideas of Witten and Kapustin in TGD framework.

1. Witten and Kapustin suggest that electric-magnetic duality and its generalization to S-duality in non-abelian is the physical counterpart of $G \leftrightarrow^L G$ duality in geometric Langlands. The model is essentially a modification $\mathcal{N} = 4$ SUSY to $\mathcal{N} = 2$ SUSY allowing this duality with Minkowski space replaced with a Cartesian product of two Riemann surfaces. In TGD framework M^4 would correspond naturally to space-time sheet allowing a slicing to string world sheets and partonic 2-surfaces. Witten and Kapustin call these 2-dimensional surfaces branes of type A and B with motivation coming from M-theory. The generalization of the basic dimensional formulas of S-duality to TGD framework implies that light-like 3-surfaces at which the signature of the induced metric changes and space-like 3-surfaces at the boundaries of CDs are analogs of brane orbits. Branes in turn would be partonic 2-surfaces. S-duality would be nothing but strong form of general coordinate invariance.

2. Witten and Kapustin introduce the notions of electric and magnetic eigen branes and formulate the duality as a transformation permuting these branes with each other. In TGD framework the obvious identification of the electric eigen branes are as string world sheets and these can be indeed identified essentially uniquely. Magnetic eigen branes would correspond to partonic 2-surfaces.

3. Witten and Kapustin introduce gauge theory with given gauge group. In TGD framework there is no need to introduce gauge theory description since the symmetry group emerges as the effective symmetry group defining measurement resolution. Gauge theory is expected to be only an approximation to TGD itself. In fact, it seems that the interpretation of G as Lie-group associated with Kac-Moody symmetry is more appropriate in TGD framework. This would mean generalization of 2-D sigma model to string model in moduli space. The action of G would not be visible in the resolution used.

4. Edward Frenkel represents the conjecture that there is mysterious 6-dimensional theory behind the geometric Langlands duality. In TGD framework this theory might correspond to twistorial formulation of quantum TGD using instead of $M^4 \times CP_2$ the space $CP_3 \times CP_3$ with space-time surfaces replaced by 6-D sphere bundles.

3.4 Finite measurement resolution realized group theoretically

The notion of finite measurement resolution allows to identify the effective symmetry groups G and ^LG in TGD framework. The most plausible interpretation of G is as Lie group giving rise to Kac-Moody type symmetry and assignable to a string model defined in moduli space of partonic 2-surfaces. By electric magnetic duality the roles of the string world sheet and partonic 2-surface can be exchanged provided the replacement $G \rightarrow G_L$ is performed. The duality means a duality of closed Euclidian strings and Minkowskian open strings.

1. The vision is that finite measurement resolution realized in terms of inclusions of HFFs corresponds to effective which is gauge or Kac-Moody type local invariance extended to quantum Yangian symmetry. A given finite measurement resolution would correspond to effective symmetry G giving rise to confinement so that the effective symmetry indeed remains invisible as finite measurement resolution requires. The finite measurement resolution should allow to emulate almost any gauge theory or string model type theory. This theory might allow super-symmetrization reducing to broken super-symmetries of quantum TGD generated by the fermionic oscillator operators at partonic 2-surfaces and string world sheets.

2. Finite measurement resolution implies that the orbit of the partonic 2-surface reduces effectively to a braid. There are two kinds of braids. Time-like braids have their ends at the boundaries of CD consisting of rational points in the intersection of real and p-adic worlds. Space-like braids are assignable to the space-like 3-surfaces at the boundaries of CD and their ends co-incide with the ends of time-like braids. The electric-magnetic duality says that the descriptions based using either kind of braids is all that is needed and that the descriptions are equivalent.

   The counterpart of $\tau \rightarrow -1/\tau$ should relate these descriptions. This need not involve transformation of effective complex Kähler coupling strength although this option cannot be excluded. If this view is correct the descriptions in terms of string world sheets and partonic 2-surfaces would correspond to electric and magnetic descriptions, which is indeed a very natural interpretation. This geometric transformation should replace G with ^LG.

3. Finite measurement resolution effectively replaces partonic 2-surface with a discrete set of points and space-time surface with string world sheets or partonic 2-surfaces. The natural question is whether finite measurement resolution also replaces geometric Langlands and the "rational" intersection of real and p-adic worlds with number theoretic Langlands and rational points of the partonic 2-surface. Notice that the rational points would be common to the string world sheets and partonic 2-surfaces so that the duality of stringy and partonic descriptions would be very natural for fintie measurement resolution.

The basic question is how the symmetry group G emerges from finite measurement resolution. Are all Lie groups possible? Here the theory of Witten and Kapustin suggests guidelines.

1. What Witten and Kapustin achieve is a transformation of a twisted $\mathcal{N} = 4$ SUSY in $M^4 = \Sigma \times C$, where Σ is "large" as compared to Riemann surface C SUSY to a sigma model in Σ with values of fields in the moduli space of Higgs bundle defined in C. If one accepts the basic conjecture that at least regions of space-time sheets allow a slicing by string world sheets and partonic 2-surfaces one indeed obtains $M^4 = \Sigma \times C$ type structure such that Σ corresponds to string world sheet and C to partonic 2-surface.

   The sigma model -or more generally string theory- would have as a natural target space the moduli space of the partonic 2-surfaces. This moduli space woudl have as coordinates its conformal moduli and the positions of the punctures expressible in terms of the imbedding space coordinates. For M^4 coordinates only the part transversal to Σ would represent physical degree of freedom and define complex coordinate. Each puncture would give rise to two complex E^2 coordinates and 2 pairs of complex CP_2 coordinates. If one identifies the string world sheets as an inverse image of a homologically non-trivial geodesic sphere as suggested in [8]. This would eliminate CP_2 coordinates as dynamical variables and one would have just n complex valued coordinates.

2. How to construct the Lie algebra of the effective symmetry group G defining the measurement resolution? If G is gauge group there is no obvious guess for the recipe. If G defines Kac-Moody

algebra the situation is much better. There exists an extremely general construction allowing a stringy construction of Kac-Moody algebra using only the elements of its Cartan algebra with central extension defined by integer valued central extension parameter k. The vertex operators defining the elements of the complement of the Cartan algebra of complexified Kac-Moody algebra are ordered exponentials of linear combinations of the Cartan algebra generators with coefficient given by the weights of the generators, which are essentially the quantum numbers assignable to them as eigenvalues of Cartan algebra generators acting in adjoint representations.

The explicit expression for the Kac-Moody generator as function of complex coordinate of Riemann sphere S^2 is

$$J_\alpha(z) =: exp(\alpha \cdot \phi(z))) : \quad .$$

$J_\alpha(z)$ represents a generator in the complement of Cartan algebra in standard Cartan basis having quantum numbers α and $\phi(z)$ represents the Cartan algebra generator allowing decomposition into positive and negative frequency parts. The weights α must have the same length $((\alpha,\alpha) = 2)$ meaning that the Lie group is simply laced. This representation corresponds to central extension parameter $k = 1$. In bosonic string models these operators are problematic since they represent tachyons but in the recent context this not a problem. The central extension parameter c for the associated Virasoro representation is also non-vanishing but this should not be a problem now.

3. What is remarkable that depending on choice of the weights α one obtains a large number of Lie algebras with same dimension of Cartan algebra. This gives excellent hopes of realizing in finite measurement resolution in terms of Kac-Moody type algebras obtained as ordered exponentials of the operators representing quantized complex E^2 coordinates. Any complexified simply laced Lie group would define a Kac-Moody group as a chacterizer of finite measurement resolution. Simply laced groups correspond by MacKay correspondence finite subgroups of $SU(2)$, which suggests that only Galois groups representable as subgroups of $SU(2)$ can be realized using this representation. It however seems that free field representations can be defined for an arbitrary affine algebra: these representations are discussed by Edward Frenkel [7].

4. The conformal moduli of the partonic 2-surface define part of the target space. Also they could play the role of conformal fields on string world sheet. The strong form of holography poses heavy constraints on these fields and the evolution of the conformal moduli could be completely fixed once their values at the ends of string world sheets at partonic 2-suraces are known. Are also the orbits of punctures fixed completely by holography from initial values for "velocities" at partonic 2-surfaces corresponding to wormhole throats at which the signature of the metric changes? If this were the case, stringy dynamics would reduce to that for point like particles defined by the punctures. This cannot be true and the natural expectation is that just the finite spatial measurement resolution allows a non-trivial stringy dynamics as quantum fluctuations below the measurement resolution.

   Could the rational values of the coordinates represent the analog of gauge choice? Or could braid ends be associated with partonic 2-surfaces which represent extrema of Kähler action in Minkowskian region giving rise to a stationary phase? Kähler action and therefore Chern-Simons action would depend on the positions of braid points. The condition that string world sheet identified as the inverse image of homologically non-trivial geodesic sphere of CP_2 [8] intersects partonic 2-surfaces at braid ends, should be enough to guarantee this. The Kähler action contains also measurement interaction term [7] but this term is not localized to braids.

5. The electric-magnetic duality induces S-duality permuting G and ${}^LG$ and the roles of string world sheet as 2-D space-time and partonic 2-surface defining defining the target manifold of string model. The moduli spaces of string world sheets and partonic 2-surfaces are in very close correspondence as implied by the strong form of holography.

3.5 How Langlands duality relates to quantum Yangian symmetry of twistor approach?

The are obvious objections against the heuristic considerations represented above.

1. One cannot restrict the attention on single partonic 2-surface or string world sheet. It is the collection of partonic 2-surfaces at the two light-like boundaries of CD and the string world sheets which define the geometric structure to which one should assign both the representations of the Galois group and the collection of world sheets as well as the groups G and ^LG. Therefore also the group G defining the measurement resolution should be assigned to the entire structure and this leaves only single option: G defines the quantum Yangian defining the symmetry of the theory. If this were not complicated enough, note that one should be also able to take into account the possibility that there are CDs within CDs.

2. The finite measurement resolution should correspond to the replacement of ordinary Lie group with something analogous to quantum group. In the simplest situation the components of quantum spinors cease to commute: as a consequence the components correlate and the dimension of the system is reduced to quantum dimension smaller than the algebraic dimension $d = 2$. Ordinary (p, q) wave mechanics is a good example about this: now the dimension of the system is reduced by a factor two from the dimension of phase space to that of configuration space.

3. Quantum Yangian algebra is indeed an algebra analogous to quantum group and according to MacKay did not receive the attention that it received as a symmetry of integrable systems because quantum groups became the industry [15]. What can one conclude about the quantum Yangian in finite measurement resolution. One can make only guesses and which can be defended only by their internal consistency.

   (a) Since the basic objects are 2-dimensional, the group G should be actually span Kac-Moody type symplectic algebra and Kac-Moody algebra associated with the isometries of the imbedding space: this conforms with the proposed picture. Frenkel has discussed the relations between affine algebras, Langlands duality, and Bethe ansatz already at previous millenium [8].

   (b) Finite measurement resolution reduces the partonic 2-surfaces to collections of braid ends. Does this mean that Lie group defining quantum Yangian group effectively reduces to something finite-dimensional? Or does the quantum Yangian property already characterize the measurement resolution as one might conclude from the previous argument? The simplest guess is that one obtains quantum Yangian containing as a factor the quantum Yangian associated with a Kac-Moody group defined by a finite-D Lie group with a Cartan algebra for which dimension equals to the total number of ends of braid strands involved. Zero energy states would be singlets for this group. This identification conforms with the general picture.

   (c) There is however an objection against the proposal. Yangian algebra contains a formal complex deformation parameter h but all deformations are equivalent to $h = 1$ deformation by a simple re-scaling of the generators labelled by non-negative integers trivial for $n = 0$ generators. Is Yangian after all unable to describe the finite measurement resolution. This problem could be circumvented by replacing Yangian with so called (twisted) quantum Yangian characterized by a complex quantum deformation parameter q. The representations of twisted quantum Yangians are discussed in [13].

   (d) The quantum Yangian group should have also as a factor the quantum Yangian assigned to the symplectic group and Kac-Moody group for isometries of H with M^4 isometries extended to the conformal group of M^4. Finite measurement resolution would be realized as a q-deformation also in these degrees of freedom.

   (e) The proposed identification looks consistent with the general picture but one can also consider a reduction of continuous Kac-Moody type algebra to its discrete version obtained by replacing partonic 2-surfaces with the ends of braid strands as an alternative.

4. The appearance of quantum deformation is not new in the context of Langlands conjecture. Frenkel has proposed Langlands correspondence for both quantum groups [11], and finite-dimensional representations of quantum affine algebras [12].

3.6 About the structure of the Yangian algebra

The attempt to understand Langlands conjecture in TGD framework led to a completely unexpected progress in the understanding of the Yangian symmetry expected to be the basic symmetry of quantum TGD and the following vision suggesting how conformal field theory could be generalized to four-dimensional context is a fruit of this work.

The structure of the Yangian algebra is quite intricate and in order to minimize confusion easily caused by my own restricted mathematical skills it is best to try to build a physical interpretation for what Yangian really is and leave the details for the mathematicians.

1. The first thing to notice is that Yangian and quantum affine algebra are two different quantum deformations of a given Lie algebra. Both rely on the notion of R-matrix inducing a swap of braid strands. R-matrix represents the projective representations of the permutation group for braid strands and possible in 2-dimensional case due to the non-commutativity of the first homotopy group for 2-dimensional spaces with punctures. The R-matrix $R_q(u,v)$ depends on complex parameter q and two complex coordinates u,v. In integrable quantum field theories in M^2 the coordinates u,v are real numbers having identification as exponentials representing Lorenz boosts. In 2-D integrable conformal field theory the coordinates u,v have interpretation as complex phases representing points of a circle. The assumption that the coordinate parameters are complex numbers is the safest one.

2. For Yangian the R-matrix is rational whereas for quantum affine algebra it is trigonometric. For the Yangian of a linear group quantum deformation parameter can be taken to be equal to one by a suitable rescaling of the generators labelled by integer by a power of the complex quantum deformation parameter q. I do not know whether this true in the general case. For the quantum affine algebra this is not possible and in TGD framework the most interesting values of the deformation parameter correspond to roots of unity.

3.6.1 Slicing of space-time sheets to partonic 2-surfaces and string world sheets

The proposal is that the preferred extremals of Kähler action are involved in an essential manner the slicing of the space-time sheets by partonic 2-surfaces and string world sheets. Also an analogous slicing of Minkowski space is assumed and there are infinite number of this kind of slicings defining what I have called Hamilton-Jaboci coordinates [2]. What is really involved is far from clear. For instance, I do not really understand whether the slicings of the space-time surfaces are purely dynamical or induced by special coordinatizations of the space-time sheets using projections to special kind of sub-manifolds of the imbedding space, or are these two type of slicings equivalent by the very property of being a preferred extremal. Therefore I can represent only what I think I understand about the situation.

1. What is needed is the slicing of space-time sheets by partonic 2-surfaces and string world sheets. The existence of this slicing is assumed for the preferred extremals of Kähler action [2]. Physically the slicing corresponds to an integrable decomposition of the tangent space of space-time surface to 2-D space representing non-physical polarizations and 2-D space representing physical polarizations and has also number theoretical meaning.

2. In zero energy ontology the complex coordinate parameters appearing in the generalized conformal fields should correspond to coordinates of the imbedding space serving also as local coordinates of the space-time surface. Problems seem to be caused by the fact that for string world sheets hyper-complex coordinate is more natural than complex coordinate. Pair of hyper-complex and complex coordinate emerge naturally as Hamilton-Jacobi coordinates for Minkowski space encountered in the attempts to understand the construction of the preferred extremals of Kähler action.

   Also the condition that the flow lines of conserved isometry currents define global coordinates lead to the to the analog of Hamilton-Jacobi coordinates for space-time sheets [2]. The physical interpretation is in terms of local polarization plane and momentum plane defined by local light-like direction. What is so nice that these coordinates are highly unique and determined dynamically.

3. Is it really necessary to use two complex coordinates in the definition of Yangian-affine conformal fields? Why not to use hyper-complex coordinate for string world sheets? Since the inverse of hyper-complex number does not exist when the hyper-complex number is light-like, hyper-complex coordinate should appear in the expansions for the Yangian generalization of conformal field as positive powers only. Intriguingly, the Yangian algebra is "one half" of the affine algebra so that only positive powers appear in the expansion. Maybe the hyper-complex expansion works and forces Yangian-affine instead of doubly affine structure. The appearance of only positive conformal weights in Yangian sector could also relate to the fact that also in conformal theories this restriction must be made.

4. It seems indeed essential that the space-time coordinates used can be regarded as imbedding space coordinates which can be fixed to a high degree by symmetries: otherwise problems with general coordinate invariance and with number theoretical universality would be encountered.

5. The slicing by partonic 2-surfaces could (but need not) be induced by the slicing of CD by parallel translates of either upper or lower boundary of CD in time direction in the rest frame of CD (time coordinate varying in the direction of the line connecting the tips of CD). These slicings are not global. Upper and lower boundaries of CD would definitely define analogs of different coordinate patches.

3.6.2 Physical interpretation of the Yangian of quantum affine algebra

What the Yangian of quantum affine algebra or more generally, its super counterpart could mean in TGD framework? The key idea is that this algebra would define a generalization of super conformal algebras of super conformal field theories as well as the generalization of super Virasoro algebra. Optimist could hope that the constructions associated with conformal algebras generalize: this includes the representation theory of super conformal and super Virasoro algebras, coset construction, and vertex operator construction in terms of free fields. One could also hope that the classification of extended conformal theories defined in this manner might be possible.

1. The Yangian of a quantum affine algebra is in question. The heuristic idea is that the two R-matrices - trigonometric and rational- are assignable to the swaps defined by space-like braidings associated with the braids at 3-D space-like ends of space-time sheets at light-like boundaries of CD and time like braidings associated with the braids at 3-D light-like surfaces connecting partonic 2-surfaces at opposite light-like boundaries of CD. Electric-magnetic duality and S-duality implied by the strong form of General Coordinate Invariance should be closely related to the presence of two R-matrices. The first guess is that rational R-matrix is assignable with the time-like braidings and trigonometric R-matrix with the space-like braidings. Here one must or course be very cautious.

2. The representation of the collection of Galois groups associated with infinite primes in terms of braided symplectic flows for braid of braids of .... braids implies that there is a hierarchy of swaps: swaps can also exchange braids of ...braids. This would suggest that at the lowest level of the braiding hierarchy the R-matrix associated with a Kac-Moody algebra permutes two braid strands which decompose to braids. There would be two different braided variants of Galois groups.

3. The Yangian of the affine Kac-Moody algebra could be seen as a 4-D generalization of the 2-D Kac-Moody algebra- that is a local algebra having representation as a power series of complex coordinates defined by the projections of the point of the space-time sheet to geodesic spheres of light-cone boundary and geodesic sphere of CP_2.

4. For the Yangian the generators would correspond to polynomials of the complex coordinate of string world sheet and for quantum affine algebra to Laurent series for the complex coordinate of partonic 2-surface. What the restriction to polynomials means is not quite clear. Witten sees Yangian as one half of Kac-Moody algebra containing only the generators having $n \geq 0$. This might mean that the positivity of conformal weight for physical states essential for the construction of the representations of Virasoro algebra would be replaced with automatic positivity of the conformal weight assignable to the Yangian coordinate.

5. Also Virasoro algebra should be replaced with the Yangian of Virasoro algebra or its quantum counterpart. This construction should generalize also to Super Virasoro algebra. A generalization of conformal field theory to a theory defined at 4-D space-time surfaces using two preferred complex coordinates made possible by surface property is highly suggestive. The generalization of conformal field theory in question would have two complex coordinates and conformal invariance associated with both of them. This would therefore reduce the situation to effectively 2-dimensional one rather than 3-dimensional: this would be nothing but the effective 2-dimensionality of quantum TGD implied by the strong form of General Coordinate Invariance.

6. This picture conforms with what the generalization of $D = 4$ $\mathcal{N} = 4$ SYM by replacing point like particles with partonic 2-surfaces would suggest: Yangian is replaced with Yangian of quantum affine algebra rather than quantum group. Note that it is the finite measurement resolution alone which brings in the quantum parameters q_1 and q_2. The finite measurement resolution might be relevant for the elimination of IR divergences.

3.6.3 How to construct the Yangian of quantum affine algebra?

The next step is to try to understand the construction of the Yangian of quantum affine algebra.

1. One starts with a given Lie group G. It could be the group of isometries of the imbedding space or subgroup of it or even the symplectic group of the light-like boundary of $CD \times CP_2$ and thus infinite-dimensional. It could be also the Lie group defining finite measurement resolution with the dimension of Cartan algebra determined by the number of braid strands.

2. The next step is to construct the affine algebra (Kac-Moody type algebra with central extension). For the group defining the measurement resolution the scalar fields assigned with the ends of braid strands could define the Cartan algebra of Kac-Moody type algebra of this group. The ordered exponentials of these generators would define the charged generators of the affine algebra.

   For the imbedding space isometries and symplectic transformations the algebra would be obtained by localizing with respect to the internal coordinates of the partonic 2-surface. Note that also a localization with respect to the light-like coordinate of light-cone boundary or light-like orbit of partonic 2-surface is possible and is strongly suggested by the effective 2-dimensionality of light-like 3-surfaces allowing extension of conformal algebra by the dependence on second real coordinate. This second coordinate should obviously correspond to the restriction of second complex coordinate to light-like 3-surface. If the space-time sheets allow slicing by partonic 2-surfaces and string world sheets this localization is possible for all 2-D partonic slices of space-time surface.

3. The next step is quantum deformation to quantum affine algebra with trigonometric R-matrix $R_{q_1}(u, v)$ associated with space-like braidings along space-like 3-surfaces along the ends of CD. u and v could correspond to the values of a preferred complex coordinate of the geodesic sphere of light-cone boundary defined by rotational symmetry. It choice would fix a preferred quantization axes for spin.

4. The last step is the construction of Yangian using rational R-matrix $R_{q_2}(u, v)$. In this case the braiding is along the light-like orbit between ends of CD. u and v would correspond to the complex coordinates of the geodesic sphere of CP_2. Now the preferred complex coordinate would fix the quantization axis of color isospin.

These arguments are of course heuristic and do not satisfy any criteria of mathematical rigor and the details could of course change under closer scrutinity. The whole point is in the attempt to understand the situation physically in all its generality.

3.6.4 How 4-D generalization of conformal invariance relates to strong form of general coordinate invariance?

The basic objections that one can rise to the extension of conformal field theory to 4-D context come from the successes of p-adic mass calculations. p-Adic thermodynamics relies heavily on the properties of partition functions for super-conformal representations. What happens when one replaces affine algebra with (quantum) Yangian of affine algebra? Ordinary Yangian involves the original algebra and its dual and from these higher multilocal generators are constructed. In the recent case the obvious interpretation for this would be that one has Kac-Moody type algebra with expansion with respect to complex coordinate w for partonic 2-surfaces and its dual algebra with expansion with respect to hyper-complex coordinate of string world sheet.

p-Adic mass calculations suggest that the use of either algebra is enough to construct single particle states. Or more precisely, local generators are enough. I have indeed proposed that the multilocal generators are relevant for the construction of bound states. Also the strong form of general coordinate invariance implying strong form of holography, effective 2-dimensionality, electric-magnetic duality and S-duality suggests the same. If one could construct the states representing elementary particles solely in terms of either algebra, there would be no danger that the results of p-adic mass calculations are lost. Note that also the necessity to restrict the conformal weights of conformal representations to be non-negative would have nice interpretation in terms of the duality.

3.7 The representation of Galois group and effective symmetry group as symplectic flow

Langlands duality involves both the Galois group and effective gauge or Kac-Moody groups G and ^LG extended to quantum Yangian and defining the automorphic forms and one should understand how these groups emerge in TGD framework.

1. What is the counterpart of Galois group in TGD? It need not be the gigantic Galois group of algebraic numbers regarded as an extension of rationals or algebraic extension of rationals. Here the proposal that infinite primes, integers and rationals are accompanied by collections of partonic 2-surfaces is very natural. Infinite primes can be mapped to irreducible polynomials of n variables and one can construct a procedure which assigns to infinite primes a collection of Galois groups. This collection of Galois groups characterizes a collection of partonic 2-surfaces.

2. How the Galois group is realized and how the symmetry group G realization finite measurement resolution is realized. How the finite-dimensional representations of Galois group lift to the finite-dimensional representations of G. The proposal is that Galois group is lifted to its braided counterpart just like braid group generalizes the symmetric group. One can speak about space-like and time-like braidings so that one would have two different kind of braidings corresponding to stringy and partonic pictures and it might be possible to understand the emergence of G and ^LG. The symplectic group for the boundary of CD define the isometries of WCW and by its infinite-dimensionality it is unique candidate for realizing representation of any group as its subgroup. he braidings are induced by symplectic flows.

3. Obviously also the symmetry groups G and ^LG should be realized as symplectic flows in appropriate moduli spaces. There are two different symplectic flows corresponding to space-like and time-like braids so that G and ^LG can be different and might differ even at the level of Lie algebra. The common realization of Galois group and symmetry group defining measurement resolution would imply Langlands duality automatically. The electric magnetic duality would in turn correspond to the possibility of two kinds of braidings. It must be emphasized that Langlands duality would be something independent of electric-magnetic duality and basically due to the realization of group representations as projective representations realized in terms of braidings. Note that also the automorphic forms define projective representations of G.

Why should the finite Galois group (possibly so!) correspond to Lie group G as it does in number theoretic Langlands correspondence?

1. The dimension of the representation of Galois group is finite and this dimension would correspond to the finite dimension for the representation of G defined by the finite-dimensional space in which G acts. This space is very naturally the moduli space of partonic 2-surfaces with n punctures corresponding to the n braid ends. A possible additional restriction is that the end points of braids are only permuted under the action of G. If the representations of the Galois group indeed automatically lift to the representations of the group defining finite measurement resolution, then Langlands duality would follow automatically.

2. The group G would correspond to the Galois group in very much the same manner as finite subgroups of $SU(2)$ correspond to simply laced Lie groups in MacKay correspondence [4]. This would generalize Mc Kay correspondence to much more general theorem holding true for the inclusions of HFFs.

An interesting open question is whether one should consider representations of the collection of Galois groups assignable to the construction of zeros for polynomials associated with infinite prime or the gigantic Galois group assignable to algebraic numbers. The latter group could allow naturally p-adic topology. The notion of finite measurement resolution would strongly suggest that one should consider the braided counterpart of the finite Galois group. This would give also a direct connection with the physics in TGD Universe. Langlands correspondence would be basic physics of TGD Universe.

3.8 The practical meaning of the geometric Langlands conjecture

This picture seems to lead naturally to number theoretic Langlands conjecture. What geometric Langlands conjecture means in TGD Universe?

1. What it means to replace the braids with entire partonic 2-surfaces. Should one keep the number of braid strands constant and allow also non-rational braid ends? What does the number of rational points correspond at WCW level? How the automorphic forms code the information about the number of rational surfaces in the intersection?

2. Quantum classical correspondence suggests that this information is represented at space-time level. Braid ends characterize partonic 2-surfaces in finite measurement resolution. The quantum state involves a quantum super position of partonic 2-surfaces with the same number of rational braid strands. Different collections of rational points are of course possible. These collections of braid ends should be transformed to each other by a discrete algebraic subgroup of the effective symmetry group G. Suppose that the orbit for a collection of n braid end points contains N different collections of braid points.

   One can construct irreps of a discrete subgroup of the symmetry group G at the orbit. Could the number N of points at the orbit define the number which could be identified as the number of rational surfaces in the intersection in the domain of definition of a given WCW spinor field defined in terms of finite measurement resolution. This would look rather natural definition and would nicely integrate number theoretic and geometric Langlands conjectures together. For infinite primes which correspond to polynomials also the Galois groups of local number fields would also entire the picture naturally.

3. One can of course consider the possibility of replacing them with light-like 3-D surfaces or space-like 3- surfaces at the ends of causal diamonds but this is not perhaps not essential since holography implies the equivalence of these identifications. The possible motivation would come from the observations that vanishing of two holomoprhic functions at the boundary of CD defines a 3-D surface.

3.9 How TGD approach differs from Witten-Kapustin approach?

The basic difference as compared to Witten-Kapustin approach [14] is that the moduli space for partonic 2-surfaces replaces in TGD framework the moduli space for Higgs field configurations. Higgs bundle defined as a holomorphich bundle together with Higgs field is the basic concept. In the simplest situations Higgs field is not a scalar but holomorphic 1-form at Riemann surface Y (analog of partonic 2-surface) related closely to the gauge potential of $M^4 = C \times Y$ whose components become scalars in spontaneous compactiifcation to C. This is in complete analogy with the fact that the values of 1-forms defining the basis of cohomology group for partonic 1-surface for cycles defining the basis of 1-homology define conformal moduli.

A possible interpretation is in terms of geometrization of all gauge fields and Higgs field in TGD framework. Color and electroweak gauge fields are geometrized in terms of projections of color Killing vectors and induced spinor connection. Conformal moduli space for the partoic 2-surface would define the geometrization for the vacuum expectation value of the Higgs field.

One can even argue that dynamical Higgs is not consistent with the notion that the modulus characterizes entire 2-surfaces. Maybe the introducing of the quantum fluctuating part of Higgs field is not appropriate. Also the fact, that for Higgs bundle Higgs is actually 1-form suggests that something might be wrong with the notion of Higgs field. Concerning Higgs the recent experimental situation at LHC is critical: it might well turn out that Higgs boson does not exist. In TGD framework the most natural option is that Higgs like particles exist but all of them are "eaten" by gauge bosons meaning that also photon, gluons possess a small mass. Something analogous to the space of Higgs vacuum expectation values might be however needed and this something could correspond to the conformal moduli space. In TGD framework the particle massivation is described in terms of p-adic thermodynamics and the dominant contribution to the mass squared comes from conformal moduli. It might be possible to interpret this contribution as an average of the contribution coming from geometrized Higgs field.

One challenge is to understand whether the moduli spaces assignable to partonic 2-surfaces and with string world sheets are so closely related that they allow the analog of mirror symmetry of the super-string models relating 6-dimensional Calabi-Yau manifolds. For Calabi-Yau:s the mirror symmetry exchanges complex and Kähler structures. Could also now something analogous make sense.

1. Strong form of general coordinate invariance and the notion of preferred extremal implies that the collection of partonic 2-surfaces fixes the collection of string world sheets (these might define single connected sheet as a connected sum). This alone suggests that there is a close correspondence between moduli spaces of the string world sheets and of partonic 2-surfaces.

2. One problem is that space-time sheets in the Minkowskian regions have hyper-complex rather than complex structure. The analog of Kähler form must represent hypercomplex imaginary unit and must be an antisymmetric form multiplied by the complex imaginary unit so that its square equals to the induced metric representing real unit.

3. How the moduli defined by integrals of complex 1-forms over cycles generalize? What one means with cycles now? How the handle numbers g_i of handles for partonic 2-surfaces reveal themselves in the homology and cohomology of the string world sheet? Do the ends of the string world sheets at the orbits of a given partonic 2-surface define curves which rotate around the handles and is the string would sheet a connected structure obtained as topological sum of this kind of string world sheets. Does the dynamics for preferred extremals of Kähler dictate this?

In the simplest situation (abelian gauge theory) the Higgs bundle corresponds to the upper half plane defined by the possible values of the inverse of the complexified coupling strength

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2} \ .$$

Does the transformation for τ defined in this manner make sense?

1. The vacuum functional is the product of exponent of imaginary Kähler action from Minkowskian regions and exponent of real Kähler action from Euclidian regions appears as an exponent proportional to this kind of parameter. The weak form of electric-magnetic duality reduces Kähler action to 3-D Chern-Simons terms at light-like wormhole throats plus possible contributions not assignable to wormhole throats. This realizes the almost topological QFT property of quantum TGD and also holography and means an enormous calculational simplification. The complexified Kähler coupling strength emerges naturally as the multiplier of Chern-Simons term if the latter contributions are not present.

2. There is however no good reason to believe that string world sheets and partonic two-surface should correspond to the values of τ and $-1/\tau$ for a moduli space somehow obtained by gluing the moduli spaces of string worlds sheets and partonic 2-surfaces. More general modular symmetries for τ seem also implausible in TGD framework. The weak form of electric magnetic duality leads to the effective complexification of gauge coupling but there is no reason to give up the idea about the quantum criticality implying quantization of Kähler coupling strength.

3. From the foregoing it is clear that the identification of G as a Kac-Moody type group extended to quantum Yangian and assignable to string model in conformal moduli space is strongly favored interpretation so that the representation of $G-^LG$ duality as a transformation of gauge coupling does not look plausible. A more plausible interpretation is as a duality between Minkowskian open string model and Euclidian closed string model with target spaces defined by corresponding moduli spaces.

4. The notion of finite measurement resolution suggesting strongly quantum group like structure is what distinguishes TGD approach from Witten's approach and from the foregoing it is clear that the identification of G as a group defining Kac-Moody type group assignable to string model in conformal moduli space and further extended to quantum Yangian is the strongly favored interpretation so that the representation of $G -^L G$ duality as a transformation of gauge coupling does not look plausible. A more plausible interpretation is as a duality between Minkowskian open string model and Euclidian closed string model with target spaces defined by corresponding moduli spaces.

5. In his lecture Edward Frenkel explains that the recent vision about the conformal moduli is as parameters analogous to gauge coupling constants. It might well be that the moduli could take the role of gauge couplings. This might allow to have a fresh view to the conjecture that the lowest three genera are in special role physically because all these Riemann surfaces are hyper-elliptic (this means global Z_2 conformal symmetry) and because for higher genera elementary particle vacuum functionals vanish for hyper-elliptic Riemann surfaces [4].

To sum up, the basic differences seem to be due to zero energy ontology, finite measurement resolution, and the identification of space-time as a 4-surface implying strong form of general coordinate invariance implying electric-magnetic and S-dualities implying also the replacement of Higgs bundle with the conformal moduli space.

4 Summary and outlook

It is good to try to see the relationship between Langlands program and TGD from a wider perspective and relate it to other TGD inspired views about problems of what I would call recent day physical mathematics. I try also to become (and remain!) conscious about possible sources of inconsistencies to see what might go wrong.

I see the attempt to understand the relation between Langlands program and TGD as a part of a bigger project the goal of which is to relate TGD to physical mathematics. The basic motivations come from the mathematical challenges of TGD and from the almost-belief that the beautiful mathematical structures of the contemporary physical mathematics must be realized in Nature somehow.

The notion of infinite prime is becoming more and more important concept of quantum TGD and also a common denominator. The infinite-dimensional symplectic group acting as the isometry group of WCW geometry and symplectic flows seems to be another common denominator. Zero energy ontology together with the notion of causal diamond is also a central concept. A further common denominator seems to be the notion of finite measurement resolution allowing discretization. Strings and super-symmetry so beautiful notions that it is difficult to imagine physics without them although super string theory has turned out to be a disappointment in this respect. In the following I mention just some examples of problems that I have discussed during this year.

Infinite primes are certainly something genuinely TGD inspired and it is reasonable to consider their possible role in physical mathematics.

1. The set theoretic view about the fundamentals of mathematics is inspired by classical physics. Cantor's view about infinite ordinals relies on set theoretic representation of ordinals and is plagued by difficulties (say Russel's paradox) [12]. Infinite primes provide an alternative to Cantor's view about infinity based on divisility alone and allowing to avoid these problems. Infinite primes are obtained by a repeated second quantization of an arithmetic quantum field theory and can be seen as a notion inspired by quantum physics. The conjecture is that quantum states in TGD Universe can be labelled by infinite primes and that standard model symmetries can be understood in terms of octonionic infinite primes defined in appropriate manner.

The replacement of ordinals with infinite primes would mean a modification of the fundamentals of physical mathematics. The physicists's view about the notion set is also much more restricted than the set theoretic view. Subsets are typically manifolds or even algebraic varieties and they allow description in terms of partial differential equations or algebraic equations.

Boolean algebra is the quintessence of mathematical logic and TGD suggests that quantum Boolean algebra should replace Boolean algebra [12]. The representation would be in terms of fermionic Fock states and in zero energy ontology fermionic parts of the state would define Boolean states of form $A \rightarrow B$. This notion might be useful for understanding the physical correlates of Boolean cognition and might also provide insights about fundamentals of physical mathematics itself. Boolean cognition must have space-time correlates and this leads to a space-time description of logical OR *resp.* AND as a generalization of trouser diagram of string models *resp.* fusion along ends of partonic 2-surfaces generalizing the 3-vertex of Feyman diagrammatics. These diagrams would give rise to fundamental logic gates.

2. Infinite primes can be represented using polynomials of several variables with rational coefficients [12]. One can solve the zeros of these polynomials iteratively. At each step one can identify a finite Galois group permuting the roots of the polynomial (algebraic function in general). The resulting Galois groups can be arranged into a hierarchy of Galois groups and the natural idea is that the Galois groups at the upper levesl act as homomorphisms of Galois groups at lower levels. A generalization of homology and cohomology theories to their non-Abelian counterparts emerges [16]: the square of the boundary operation yields unit element in normal homology but now an element in commutator group so that abelianization yields ordinary homology. The proposal is that the roots are represented as punctures of the partonic 2-surfaces and that braids represent symplectic flows representing the braided counterpars of the Galois groups. Braids of braids of.... braids structrure of braids is inherited from the hierarchical structure of infinite primes.

   That braided Galois groups would have a representation as symplectic flows is exactly what physics as generalized number theory vision suggests and is applied also to understand Langlands conjectures. Langlands program would be modified in TGD framework to the study of the complexes of Galois groups associated with infinite primes and integers and have direct physical meaning.

The notion of finite measurement resolution realized at quantum level as inclusions of hyper-finite factors and at space-time level in terms of braids replacing the orbits of partonic 2-surfaces - is also a purely TGD inspired notion and gives good hopes about calculable theory.

1. The notion of finite measurement resolution leads to a rational discretization needed by both the number theoretic and geometric Langlands conjecture. The simplest manner to understand the discretization is in terms of extrema of Chern-Simons action if they correspond to "rational" surfaces. The guess that the rational surfaces are dense in the WCW just as rationals are dense in various number fields is probably quite too optimistic physically. Algebraic partonic 2-surfaces containin typically finite number of rational points having interpretation in terms of finite measurement resolution. Same might apply to algebraic surfaces as points of WCW in given quantum state.

2. The charged generators of the Kac-Moody algebra associated with the Lie group G defining measurement resolution correspond to tachyonic momenta in free field reprsesentation using ordered exponentials. This raises unpleasant question. One should have also a realization for the coset construction in which Kac-Moody variant of the symplectic group of $\delta M^4_{\pm}$ and Kac-Moody algebra of isometry group of H assignable to the light-like 3-surfaces (isometries at the level of WCW *resp.* H) define a coset representation. Equivalence Principle generalizes to the condition that the actions of corresponding super Virasoro algebras are identical. Now the momenta are however non-tachyonic.

   How these Kac-Moody type algebras relate? From p-adic mass calculations it is clear that the ground states of super-conformal representations have tachyonic conformal weights. Does this mean that the ground states can be organized into representations of the Kac-Moody algebra representing finite measurement resolution? Or are the two Kac-Moody algebra like structures completely independent. This would mean that the positions of punctures cannot correspond to the H-coordinates appearing as arguments of sympletic and Kac-Moody algebra giving rise to Equivalence Principle. The fact that the groups associated with algebras are different would allow this.

TGD is a generalization of string models obtained by replacing strings with 3-surfaces. Therefore it is not surprising that stringy structures should appear also in TGD Universe and the strong form of general coordinate invariance indeed implies this. As a matter fact, string like objects appear also in various applications of TGD: consider only the notions of cosmic string [5] and nuclear string [10]. Magnetic flux tubes central in TGD inspired quantum biology making possible topological quantum computation [6] represent a further example.

1. What distinguishes TGD approach from Witten's approach is that twisted SUSY is replaced by string model like theory with strings moving in the moduli space for partonic 2-surfaces or string world sheets related by electric-magnetic duality. Higgs bundle is replaced with the moduli space for punctured partonic 2-surfaces and its electric dual for string world sheets. The new element is the possibility of trouser vertices and generalization of 3-vertex if Feynman diagrams having interpretation in terms of quantum Boolean algebra.

2. Stringy view means that all topologies of partonic 2-surfaces are allowed and that also quantum superpositions of different topologies are allowed. The restriction to single topology and fixed moduli would mean sigma model. Stringy picture requires quantum superposition of different moduli and genera and this is what one expects on physical grounds. The model for CKM mixing indeed assumes that CKM mixing results from different topological mixings for U and D type quarks [11] and leads to the notion of elementary particle vacuum functional identifiable as a particular automorphic form [4].

3. The twisted variant of $\mathcal{N} = 4$ SUSY appears as TQFT in many mathematical applications proposed by Witten and is replaced in TGD framework by the stringy picture. Supersymmetry would naturally correspond to the fermionic oscillator operator algebra assignable to the partonic 2-surfaces or string world sheet and SUSY would be broken.

When I look what I have written about various topics during this year I find that symplectic invariance and symplectic flows appear repeatedly.

1. Khovanov homology provides very general knot invariants. In [?] rephrased Witten's formulation about Khovanov homology as TQFT in TGD framework. Witten's TQFT is obtained by twisting a 4-dimensional $\mathcal{N} = 4$ SYM. This approach generalizes the original 3-D Chern-Simons approach of Witten. Witten applies twisted 4-D $\mathcal{N} = 4$ SYM also to geometric Langlands program and to Floer homology.

   TGD is an almost topological QFT so that the natural expectation is that it yields as a side product knot invariants, invariants for braiding of knots, and perhaps even invariants for 2-knots: here the dimension $D = 4$ for space-time surface is crucial. One outcome is a generalization of the notion of Wilson loop to its 2-D variant defined by string world sheetw and a unique identification of string world sheet for a given space-time surface. The duality between the descriptions based on string world sheets and partonic 2-surfaces is central. I have not yet discussed the implications of the conjectures inpired by Langlands program for the TGD inspired view about knots.

2. Floer homology generalizes the usual Morse theory and is one of the applications of topological QFTs discussed by Witten using twisted SYM. One studies symplectic flows and the basic objects are what might regarded as string world sheets referred to as pseudo-holomorphic surfaces. It is now wonder that also here TGD as almost topological QFT view leads to a generalization of the QFT vision about Floer homology [16]. The new result from TGD point of view was the realization that the naivest possible interpretation for Kähler action for a preferred extremal is correct. The contribution to Kähler action from Minkowskian regions of space-time surface is imaginary and has identification as Morse function whereas Euclidian regions give the real contribution having interpretation as Kähler function. Both contributions reduce to 3-D Chern-Simons terms and under certain additional assumptions only the wormhole throats at which the signature of the induced metric changes from Minkowskian to Euclidian contribute.

3. Gromov-Witten invariants are closely related to Floer homology and their definition involves quantum cohomology in which the notion of intersection for two varieties is more general taking into account "quantum fuzzines". The stringy trouser vertex represent the basic diagram: the incoming string world sheets intersect because they can fuse to single string world sheet. Amazingy, this is just that OR in quantum Boolean algebra suggested by TGD. Another diagram would be AND responsible for genuine particle reactions in TGD framework. There would be a direct connection with quantum Boolean algebra.

Number theoretical universality is one of the corner stones of the vision about physics as generalized number theory. One might perhaps say that a similar vision has guided Grothendieck and his followers.

1. The realization of this vision involves several challenges. One of them is definition of p-adic integration. At least integration in the sense of comology is needed and one might also hope that numerical approach to integration exists. It came as a surprise to me that something very similar to number theoretical universality has inspired also mathematicians and that there exist refined theories inspired by the notion of motive introduced by Groethendieck to to define universal cohomology applying in all number fields. One application and also motivation for taking motives very seriously is notivic integration which has found applications in in physics as a manner to calculate twistor space integrals defining scattering amplitudes in twistor approach to $\mathcal{N} = 4$ SUSY. The essence of motivic integral is that integral is an algebraic operation rather than defined by a measure. One ends up with notions like scissor group and integration as processing of symbols. This is of course in spirit with number theoretical approach where integral as measure is replaced with algebraic operation. The problem is that numerics made possible by measure seems to be lost.

2. The TGD inspired proposal for the definition of p-adic integral relies on number theoretical universality reducing the integral essentially to integral in the rational intersection of real and p-adic worlds. An essential role is played at the level of WCW by the decomposition of WCW to a union of symmetric spaces allowing to define what the p-adic variant of WCW is. Also this would conform with the vision that infinite-dimensional geometric existence is unique just from the requirement that it exists. One can consider also the possibility of having p-adic variant of numerical integration [16].

Twistor approach has led to the emergence of motives to physics and twistor approach is also what gives hopes that some day quantum TGD could be formulated in terms of explicit Feynman rules or their twistorial generalization [13, 15].

1. The Yangian symmetry and its quantum counterpart were discovered first in integrable quantum theories is responsible for the success fo the twistorial approach. What distinguishes Yangian symmetry from standard symmetries is that the generators of Lie algebra are multilocal. Yangian symmetry is generalized in TGD framework since point like particles are replaced by partonic 2-surfaces meaning that Lie group is replaced with Kac-Moody group or its generalization. Finite measurement resolution however replaces them with discrete set of points definining braid strands so that a close connection with twistor approach and ordinary Yangian symmetry is suggestive in finite measurement resolution. Also the fact that Yangian symmetry relates closely to topological string models supports the expectation that the proposed stringy view about quantum TGD could allow to formulate twistorial approach to TGD.

2. The vision about finite measurement resolution represented in terms of effective Kac-Moody algebra defined by a group with dimension of Cartan algebra given by the number of braid strands must be consistent with the twistorial picture based on Yangians and this requires extension to Yangian algebra- as a matter to quantum Yangian. In this picture one cannot speak about single partonic 2-surface alone and the same is true about the TGD based generalization of Langlands probram. Collections of two-surfaces and possibly also string world sheets are always involved. Multilocality is also required by the basic properties of quantum states in zero energy ontology.

3. The Kac-Moody group extended to quantum Yangian and defining finite measurement resolution would naturally correspond to the gauge group of $\mathcal{N} = \triangle$ SUSY and braid points to the arguments of N-point functions. The new element would be representation of massive particles

as bound states of massles particles giving hopes about cancellation of IR divergences and about exact Yangian symmetry. Second new element would be that virtual particles correspond to wormholes for which throats are massless but can have different momenta and opposite signs of energies. This implies that absence of UV divergences and gives hopes that the number of Feynman diagrams is effectively finite and that there is simple expression of twistorial diagrams in terms of Feynman diagrams [15].

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