Long length scale limit of TGD as General Relativity with sub-manifold constraint

What is the precise relationship of the long length scale limit of TGD to General Relativity as a description of gravitational interactions? On basis of physical intuition it is clear that Einstein's equations hold true for the matter topologically condensed around vacuum extremals of Kähler action and that energy momentum tensor can be described as average description for small deformations of vacuum extremals. The question is what happens in case of non-vacuum extremals. Does a simple variational principle leading to Einstein's equations at long length scale limit exist and allow to identify the solutions as extremals of Kähler action?

The answer to the question is affirmative. It has been clear from the beginning that TGD in long length scales as a theory of gravitational interactions is General Relativity with a sub-manifold constraint. The problem is to formulate this statement so that extremals of Kähler action are consistent with Einstein's equations.
Consider first a simpler situation for which Kähler action is replaced with four-volume.

1. Let us start from an action containing curvature scalar and a part describing matter. The simplest that one can try is to add just a constraint term

$$\Lambda^{\alpha\beta}(g_{\alpha\beta} - h_{kl}\partial_\alpha h^k \partial_\beta h^l) \tag{1}$$

   telling that the metric is induced metric.

2. The resulting gravitational field equations obtained by varying with respect to $g_{\alpha\beta}$ would be Einstein equations $T^{\alpha\beta} - kG^{\alpha\beta} = 0$ modified to

$$T^{\alpha\beta} - kG^{\alpha\beta} = \Lambda^{\alpha\beta} \ . \tag{2}$$

   Einstein's equations would be modified by the vacuum energy term which satisfies an additional constraint equation following from the variation with respect to imbedding space coordinates.

3. The variation with respect to imbedding space coordinates gives

$$D_\beta(\Lambda^{\alpha\beta}\partial_\beta h^l) = 0 \ . \tag{3}$$

   The latter equation is satisfied if space-time surface is an extremal of general coordinate invariant action constructed from the induced metric only. Volume term would be the simplest possibility and this would give

$$\Lambda^{\alpha\beta} = Kg^{\alpha\beta} \ ,$$

   and Eq. 2 would give Einstein's equations with cosmological constant. One can get rid of cosmological constant simply by adding to the curvature scalar part cosmological term compensating it. It is essential that the energy momentum current $T^{\alpha k} = Kg^{\alpha\beta}\partial_\beta h^k$ is parallel to the space-time surfaces as an imbedding space-vector field: this is true for actions involving only the induced metric.

In the case Kähler action both induced metric and induced Kähler form appear as field variables expressible in terms of the imbedding space coordinates. The energy momentum currents $T_k^\alpha = \partial L_K/\partial_\alpha h^k$ appearing in the field equations for Kähler action contains a part orthogonal to the space-time surface so that one cannot have

$$T^{\alpha k} = T^{\alpha\beta}\partial_\beta h^k$$

since the right hand side is parallel to the space-time surface. This makes the situation more complex.

1. One can express the sub-manifold constraint using the projections of vielbein of H rather than metric so that one obtains the constraint term

$$\Lambda^{\alpha A}(e_{A\alpha} - e_{Ak}\partial_\alpha h^k) \ . \tag{4}$$

2. Besides this the action contains the constraint terms

$$\begin{aligned} &\Lambda^{\alpha\beta}(g_{\alpha\beta} - e_{A\alpha}e^A_\beta) \ , \\ &F^{\alpha\beta}(J_{\alpha\beta} - J_{AB}e^A_\alpha e^B_\beta) \end{aligned} \tag{5}$$

   with an obvious interpretation.

3. One must also add to the action Kähler action density

$$L_K = \frac{1}{2g_K^2}J^{\alpha\beta}J_{\alpha\beta}\sqrt{g} \ , \tag{6}$$

   where $J_{\alpha\beta}$ is treated as as a primary gauge field in the variation.

The resulting field equations give field equations for an extremal of Kähler action and Einstein's equations.

1. The gravitational field equations are obtained by varying with respect to $g_{\alpha\beta}$ regarded as a primary field

$$\Lambda^{\alpha\beta} = T^{\alpha\beta} - kG^{\alpha\beta} + T_K^{\alpha\beta} \ . \tag{7}$$

   Here $T_K^{\alpha\beta}$ is standard energy momentum tensor associated with Kähler action treating $J_{\alpha\beta}$ as a primary field.

2. The variation with respect to $J_{\alpha\beta}$ regarded as a primary field gives

$$F^{\alpha\beta} = J^{\alpha\beta}\sqrt{g} \ . \tag{8}$$

3. The variation with respect to $e_{A\alpha}$ gives

$$2\Lambda^{\alpha\beta}e^A_\beta + J^{\alpha\beta}J^{AB}e_{B\beta} + \Lambda^{A\alpha} \quad = \quad 0 \ . \tag{9}$$

4. Finally, the variation with respect to h^k gives

$$D_\beta(\Lambda^{A\alpha}e^k_A) \quad = \quad 0 \ . \tag{10}$$

   These equations require a variational principle and are equivalent with those for the extremals of Kähler action if one make the identification

$$\Lambda^{A\alpha} \quad = \quad e^A_k T^{\alpha k} \ \ , \ \ T^{\alpha k} = \frac{\partial L_K}{\partial_\alpha h^k} \ . \tag{11}$$

5. Substituting $\Lambda^{\alpha\beta}$ as given by Eq. 11 and $\Lambda^{\alpha\beta}$ as given by Eq. 7 to Eq. 9, one finds that the terms involving Kähler gauge field cancel each other neatly, and one obtains

$$(T^{\alpha\beta} - kG^{\alpha\beta})e^A_\beta \quad = \quad 0 \ . \tag{12}$$

   Which is equivalent with Einstein's equations. Note that the addition of Kähler action is necessary in order to compensate the terms orthogonal to the space-time surface and -somewhat paradoxically- implies that Kähler action does not contribute to the energy momentum tensor. This is as it should be.