Trying to fuse the basic mathematical ideas of quantum TGD to a single coherent whole

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Abstract

The theoretical framework behind TGD involves several different strands and the goal is to unify them to a single coherent whole. TGD involves number theoretic and geometric visions about physics and M^8-H duality, analogous to Langlands duality, is proposed to unify them. Also quantum classical correspondence (QCC) is a central aspect of TGD. One should understand both the M^8-H duality and QCC at the level of detail.

The following mathematical notions are expected to be of relevance for this goal.

1. Von Neumann algebras, call them M, in particular hyperfinite factors of type II_1 (HFFs), are in a central role. A both the geometric and number theoretic side, QCC could mathematically correspond to the relationship between M and its commutant M'.

   For instance, symplectic transformations leave induced Kähler form invariant and various fluxes of Kähler form are symplectic invariants and correspond to classical physics commuting with quantum physics coded by the super symplectic algebra (SSA). On the number theoretic side, the Galois invariants assignable to the polynomials determining space-time surfaces are analogous classical invariants.

2. The generalization of ordinary arithmetics to quantum arithmetics obtained by replacing $+$ and $\times$ with $\oplus$ and $\otimes$ allows us to replace the notions of finite and p-adic number fields with their quantum variants. The same applies to various algebras.

3. Number theoretic vision leads to adelic physics involving a fusion of various p-adic physics and real physics and to hierarchies of extensions of rationals involving hierarchies of Galois groups involving inclusions of normal subgroups. The notion of adele can be generalized by replacing various p-adic number fields with the p-adic representations of various algebras.

4. The physical interpretation of the notion of infinite prime has remained elusive although a formal interpretation in terms of a repeated quantization of a supersymmetric arithmetic QFT is highly suggestive. One can also generalize infinite primes to their quantum variants. The proposal is that the hierarchy of infinite primes generalizes the notion of adele.

The formulation of physics as Kähler geometry of the "world of classical worlds" (WCW) involves of 3 kinds of algebras A; supersymplectic isometries SSA acting on $\delta M^4_+ \times CP_2$ SSA_n, affine algebras Aff acting on light-like partonic orbits, and isometries I of light-cone boundary δM^4_+, allowing hierarchies hierarchies A_n.

The braided Galois group algebras at the number theory side and algebras $\{A_n\}$ at the geometric side define excellent candidates for inclusion hierarchies of HFFs. M^8-H duality

suggests that n corresponds to the degree $n$of the polynomial P defining space-time surface and that the n roots of P correspond to n braid strands at H side. Braided Galois group would act in A_n and hierarchies of Galois groups would induce hierarchies of inclusions of HFFs. The ramified primes of P would correspond to physically preferred p-adic primes in the adelic structure formed by p-adic variants of A_n with $+$ and $\times$ replaced with $\oplus$ and $\otimes$.

Contents

1 Introduction

I have had a very interesting discussions with Baba Ilya Iyo Azza about von Neumann algebras [A10]. I have a background of physicist and have suffered a lot of frustration in trying to understand hyperfinite factors of type II_1 (HFFs, `https://cutt.ly/OX8uP32`) by trying to read mathematicians' articles.

I cannot understand without a physical interpretation and associations to my own big vision TGD. Again I stared at the basic definitions, ideas and concepts trying to build a physical interpretation. This is not my first attempt to understand the possible role of HFFs in TGD: I have written already earlier of the possible role of von Neumann algebras in the TGD framework [K12, K9]. In the sequel I try to summarize what I have possibly understood with my meager technical background.

In the first section I will redescribe the basic notions and ideas related to von Neumann algebras as I see them now, in particular HFFs, which seem to be especially relevant for TGD because of their "hyperfiniteness" property implying that they are effectively finite-D matrix algebras.

There are also more general factors of type II_1, in particular those related to the notion of free probability (`hhttps://cutt.ly/SX2ftyx`), which is a notion related to a theory of non-commutative random variables. The free group generated by a finite number of generators is basic notion and the group algebras associated with free groups are factors of type II_1. The isomorphism problem asks whether these algebras are isomorphic for different numbers of generators. These algebras are not hyperfinite and from the physics point of view this is not a good news.

1.1 Basic notions of HFFs from TGD perspective

In this section I will describe my recent, still rather primitive physicist's understanding of HFFs. Factor M and its commutant M' are central notions in the theory of von Neumann algebras. An important question, not discussed earlier, concerns the physical counterparts of M and M'. I will not discuss technical details: I have made at least a noble attempt to do this earlier [K12, K9].

1. In the TGD framework, one can distinguish between quantum degrees of freedom and classical ones, and classical physics can be said to be an exact part of quantum physics.

2. The formulation of physics as Kähler geometry of the "world of classical worlds" (WCW) involves hierarchies A_n of 3 kinds of algebras; supersymplectic algebras acting on $\delta M^4_+ \times CP_2$ SSA_n assumed to induce isometries of WCW, affine algebras Aff_n associated with isometries and holonomies of $H = M^4 \times CP_2$ acting on light-like partonic orbits, and isometries I_n of the light-cone boundary δM^4_+.

   At the H-side, quantum degrees of freedom are assignable to A_n, which would correspond to M.

   In zero energy ontology (ZEO) [K13] states are quantum superpositions of preferred extremals. Preferred extremals depend on zero modes, which are symplectic invariants and do not appear in the line element of WCW. Zero modes serve as classical variables, which commute with super symplectic transformations and could correspond to M' for SSA_n at H-side. Similar identification of analogs of zero modes should be possible for Aff_n and I_n.

3. In the number theoretic sector at the M^8-side, braided group algebras would correspond to quantum degrees of freedom, that is M. M' would correspond to some number theoretic invariants of polynomials P determining the space-time surface in H by $M^8 - H$ duality [L11, L12]. The set of roots of P and ramified primes dividing the discriminant of P are such invariants.

1.2 Bird's eye view of HFFs in TGD

A rough bird's eye view of HFFs is discussed with an emphasis on their physical interpretation. There are two visions of TGD: the number theoretic view [L5, L6] and the geometric view [K2, K1, K5, K7] and $M^8 - H$ duality relates these views [L18, L11, L12].

1. At the M^8 side, the p-adic representations of braided group algebras of Galois groups associated with hierarchies of extensions of rationals define natural candidates for the inclusion hierarchies of HFFs.

   Braid groups represent basically permutations of tensor factors and the same applies to the braided Galois groups with S_n restricted to the Galois group.

   A good guess is that braid strands correspond to the roots of a polynomial labelling mass shells H^3 in $M^4 \subset M^8$. The 3-D mass shells define a 4-surface in M^8 by holography based on associativity, which makes possible holography. The condition that the normal space of the 4-surface in M^8 is associative and contains 2-D commutative sub-space guarantees holography both holography and $M^8 - H$ duality mapping this 4-surface to space-time surface in H.

2. At the H side there are 3 algebras.

   (a) The subalgebras SSA_n of super-symplectic algebra (SSA) are assumed to induce isometries of WCW. Since SSA and also other algebras have non-negative conformal weights, it has a hierarchy of subalgebras SSA_n with conformal weights coming as n-multiples of those for SSA.

   (b) There are also affine algebras Aff associated with H isometries acting on light-like orbits of partonic 2-surfaces and having similar hierarchy of Aff_n. Both isometries and holonomies of H are involved.

   (c) Light-cone boundary allows infinite dimensional isometry group I consisting of generalized conformal transformation combined with a local scaling allowing similar hierarchy I_n.

   One should understand how the number theoretic and geometric hierarchies relate to each other and a good guess is that braided group algebras act on braids assignable to SSA_n with n interpreted as the number of braid strands and thus the degree n of P.

Also the interpretational problems related to quantum measurement theory and probability interpretation are discussed from the TGD point of view, in which zero energy ontology (ZEO) allows us to solve the basic problem of quantum measurement theory.

1.3 $M^8 - H$ duality and HFFS

$M^8 - H$ duality [L11, L12] suggests that the hierarchies of extensions of rationals at the number theoretic side and hierarchies of HFFs at the geometric side are closely related.

The key idea is that the braided Galois groups at M^8-side interact on algebras $A_n \in \{SSA_n, Aff_n, I_n\}$ at H level as number theoretic braid groups permuting the tensor factors assignable to the braid strands, which correspond to the roots of the polynomial P.

The basic notions associated with a polynomial P with rational coefficients having degree n are its n roots, ramified primes as factors of the discriminant defined by the difference of its roots, and Galois group plus a set of Galois invariants such as symmetric polynomials of roots. The Galois group is the same for a very large number of polynomials P. The question concerns the counterparts of these notions at the level of H?

An educated guess is that the n roots of P label the strands of an n-braid in H assignable to A_n, ramified primes correspond to physically preferred p-adic primes in the adelic structure formed by various p-adic representations $A_{n,p}$ of the algebras A_n and the Galois group algebra associated with the polynomial P with degree n.

This picture suggests a generalization of arithmetics to quantum arithmetics based on the replacement of $+$ and $\times$ with $\oplus$ and $\otimes$ and replacement of numbers with representations of groups or algebras [L23]. This implies a generalization of adele by replacing p-adic numbers with the p-adic quantum counterparts of algebras A_n.

The mysterious McKay correspondence [A13] has inspired several articles during years [L4, L10, L9, L23] but it is fair to say that I do not really understand it. Hence I could not avoid the temptation to attack this mystery also in this article.

1.4 Infinite primes

The notion of infinite primes [K11, K6] is one of the ideas inspired by TGD, which has waited for a long time for its application. Their construction is analogous to a quantization of supersymmetric arithmetic quantum field theory.

1. The analog of Dirac sea X is defined by the product of finite primes and one "kicks" from sea a subset of primes defining a square free integer n_F to get the sum $X/n_F + n_F$. One can also add bosons to X/n_F *resp.* n_F multiplying it with integer n_{B_1} *resp.* n_{B_2}, which is divisible only by primes dividing Z/n_F *resp.* n_F.

2. This construction generalizes and one can form polynomials of X to get infinite primes analogous to bound states. One can consider instead of $P(X)$ a polynomial $P(X,Y)$, where Y is the product of all primes at the first level thus involving the product of all infinite primes already constructed, and repeat the procedure. One can repeat the procedure indefinitely and the formal interpretation is as a repeated quantization. The interpretation could be in terms of many-sheeted space-time or abstraction process involving formation of logical statements about statements about ...

3. The polynomials Q could also be interpreted as ordinary polynomials. If $Q(X) = P(X)$, where $P(X)$ is the polynomial defining a 4-surface in M^8, the space-time surface X^4 in H would correspond to infinite prime. This would give a "quantization" of P defining the space-time surface.

   The polynomial P defining 4-surface in H would fix various quantum algebras associated with it. The polynomials $P(X_1, X_2, ...X_n)$ could be interpreted as $n-1$-parameter families defining surfaces in the "world of classical worlds" (WCW) [L18] (for the development of the notion see [K2, K1, K5, K7]).

4. X is analogous to adele and infinite primes could be perhaps seen as a generalization of the notion of adele. One could assign p-adic variants of various HFFs to the primes defining the adele and $+$ and $\times$ could be replaced with $\oplus$ and $\otimes$. The physical interpretation of ramified primes of P is highly interesting.

In the last section, I try to guess how the fusion of these building blocks by using the ideas introduced in the previous sections could give rise to what might be called quantum TGD. It must be made clear that the twistor lift of TGD [L20, L21] is not considered in this work.

2 Basic notions related to hyperfinite factors of type II_1 from TGD point of view

In this section, the basic notions of hyperfinite factors (HFFs) as a physicists from the TGD point of view will be discussed. I have considered HFFs earlier several times [K12, K9] and will not discuss here the technical details of various notions.

2.1 Basic concepts related to von Neumann algebras

John von Neumann proposed that the algebras, which now carry his name are central for quantum theory [A10]. Von Neumann algebra decomposes to a direct integral of factors appearing and there are 3 types of factors corresponding to types I, II, and III.

2.1.1 Inclusion/embedding as a basic aspect of physics

Inclusion (`https://cutt.ly/NX8eWwa`, `https://cutt.ly/cX8eUuf`, `https://cutt.ly/4X8ePn6`)) is a central notion in the theory of factors. Inclusion/imbedding involving induction of various geometric structures is a key element of classical and quantum TGD.

One starts from the algebra $B(H)$ of bounded operators in Hilbert space. This algebra has naturally hermitian conjugation $*$ as an antiunitary operation and therefore one talks of C_* algebras. von Neumann algebra is a subalgebra of $B(H)$. Already here an analog of inclusion is involved (https://cutt.ly/3XkPO2s). There are also inclusions between von Neumann algebras, in particular HFFs.

What could the inclusion of von Neumann algebra to $B(H)$ as subalgebra mean physically? In the TGD framework, one can identify several analogies.

1. Space-time is a 4-surface in $H = M^4 \times CP_2$: analog of inclusion reducing degrees of freedom.

2. Space-time is not only an extremal of an action [K8] [L19] but a preferred extremal (PE), which satisfies holography so that it is almost uniquely defined by a 3-surface. This guarantees general coordinate invariance at the level of H without path integral. I talk about preferred extremals (PEs) analogous to Bohr orbits. Space-time surface as PE is a 4-D minimal surface with singularities [L19]: there is an analogy with a soap film spanned by frames. This implies a small failure of determinism localizable at the analogs of frames so that holography is not completely unique.

   Holography means that very few extremals are physically possible. This Bohr orbit property conforms with the Uncertainty Principle. Also HFFs correspond to small sub-spaces of $B(H)$. Quantum classical correspondence suggests that this analogy is not accidental.

2.1.2 The notion of commutant and its physical interpretation in the TGD framework

The notion of the commutant M' of M, which also defines HFF, is also essential. What could be the physical interpretation of M'? TGD suggests 3 important hierarchies of HFFs as algebras A_n. A_n could correspond to super-symplectic algebras SSA_n acting at $\delta M^4_- \times CP_2$; to an affine algebras Aff_n acting at the light-like partonic orbits; or to an isometry algebra I_n acting at δM^4_+. All these HFF candidates have commutants and would have interpretation in terms of quantum-classical correspondence.

One can consider SSA as an example.

1. In TGD, one has indeed an excellent candidate for the commutant. Supersymplectic symmetry algebra (SSA) of $\delta M^4_+ \times CP_2$ (δM^4_+ denotes the boundary of a future directed light-cone) is proposed to act as isometries of the "world of classical worlds" (WCW) consisting of space-time surfaces as PEs (very, very roughly).

   Symplectic symmetries would be generated by Hamiltonians, which are products of Hamiltonians associated with δM^4_+ (metrically sphere S^2) and CP_2. Symplectic symmetries are conjectured to act as isometries of WCW and gamma matrices of WCW extend symplectic symmetries to super-symplectic ones.

   Hamiltonians and their super-counterparts generate the super-symplectic algebra (SSA) and quantum states are created by using them. SSA is a candidate for HFF. Call it M. What about M?

2. The symplectic symmetries leave invariant the induced Kähler forms of CP_2 and contact form of δM^4_+ (assignable to the analog of Kähler structure in M^4).

3. The wave functions in WCW depending of magnetic fluxes defined by these Kähler forms over 2-surfaces are physically observables which commute SSA and with M. These fluxes are in a central role in the classical view about TGD and define what might perhaps be regarded as a dual description necessary to interpret quantum measurements.

   Could M' correspond or at least include the WCW wave functions (actually the scalar parts multiplying WCW spinor fields with WCW spinor for a given 4-surface a fermionic Fock state) depending on these fluxes only? I have previously talked of these degrees of freedom as zero modes commuting with quantum degrees of freedom and of quantum classical correspondence.

4. There is $M - M'$ correspondence also for number theoretic degrees of freedom, which naturally appear from the number theoretic M^8 description mapped to H-description. Polynomials P associated with a given Galois group are analogous to symplectic degrees of freedom with given fluxes as symplectic invariants. Galois groups and Galois invariants are "classical" invariants at the M^8 side and should have counterparts on the H side. For instance, the degree n of polynomial P could correspond to the number of braid stran

2.1.3 More algebraic notions

There are further algebraic notions involved. The article of John Baez (`https://cutt.ly/VXlQyqD`) describes these notions nicely.

1. The condition $M'' = M$ is a defining algebraic condition for von Neumann algebras. What does this mean? Or what could its failure mean? Could M'' be larger than M? It would seem that this condition is achieved by replacing M with M''.

   $M'' = M$ codes algebraically the notion of weak continuity, which is motivated by the idea that functions of operators obtained by replacing classical observable by its quantum counterpart are also observables. This requires the notion of continuity. Every sequence of operators must approach an operator belonging to the von Neumann algebra and this can be required in a weak sense, that is for matrix elements of the operators.

   Does $M'' = M$ mean that the classical descriptions and quantum descriptions are somehow equivalent? At first, this looks nonsensical but when one notices that the scalar parts of WCW spinor fields correspond to wave functions in the zero mode of WCW which do not appear in the line element of WCW, this idea starts to look more sensible. In quantum measurements the outcome is indeed expressed in terms of classical variables. Zero modes and quantum fluctuating modes would provide dual descriptions of physics.

2. There is also the notion of hermitian conjugation defined by an antiunitary operator J: $a^\dagger = JAJ$. This operator is absolutely essential in quantum theory and in the TGD framework it is geometrized in terms of the Kähler form of WCW. The idea is that entire quantum theory, rather than only gravitation or gravitation and gauge interactions should be geometrized. Left multiplication by JaJ corresponds to right multiplication by a.

2.1.4 Connes tensor product and category theoretic notions

Connes tensor product (Connes fusion) [A2] appears in the construction of the hierarchy of inclusions of HFFs. For instance, matrix multiplication has an interpretation as Connes tensor product reduct tensor product of matrices to a matrix product. The number of degrees of freedom is reduced. The tensor product $A \otimes_R B$ depends on the coefficient ring R acting as right multiplication in A and left multiplication in B. If the dimension of R increases, the dimension of A (B) as a left/right R module is reduced. For instance, A as an A-module is 1-dimensional.

Also category theory related algebraic notions appear. I still do not have an intuitive grasp about category theory. In any case, one would have a so-called 2-category (`https://cutt.ly/3XkPO2s`). M and N correspond to 0-morphisms (objects). One can multiply arguments of functions in $L^2(M)$ and $L^2(N)$ by M or N.

Bimodule (`https://cutt.ly/EX885WA`) is a key notion. For instance the set of $R_{m,n}$ of $m \times n$ matrices is a bimodule, which is a left (right) module with respect to $m \times m$ ($n \times n$) matrices. One can replace matrices with algebras. The bimodule $_MM_M$ *resp.* $_NN_N$ is analogous to $m \times m$ *resp.* $n \times n$ matrices. They correspond to 1-morphisms, which behave like units. The bimodule $_MN_N$ *resp.* $_NN_M$ is analogous to $m \times n$ *resp.* $n \times m$ matrices. These two bimodules correspond to a generating 1-morphisms mapping N to M *resp.* M to N. Bimodule map corresponds to 2-morphisms. Connes tensor product defines what category theorists call a tensor functory.

2.1.5 The notions of factor and trace

The notion of factor as a building block of more complex structures is central and analogous to the notion of simple group or prime. Factor is a von Neumann algebra, which is simple in the sense that it has a trivial center consisting of multiples of unit operators. The algebra is direct sum or integral over different factors.

The notion of trace is fundamental and highly counter intuitive. For the factors of type I, it is just the ordinary trace and the trace $Tr(I)$ of the unit operator is equal to the dimension n of the Hilbert space. This notion is natural when direct sum is the key notion. For the other factors, the situation is different.

Factors can be classified into three types: $I, II, and III$.

1. For factors of type I associated with three bosons, the trace equals n in the n-D case and ∞ in the infinite-D case.

2. A highly non-intuitive and non-trivial axiom relating to HFFs as hyperfinite factors of type II_1 is that the trace of the unit operator satisfies $Tr(Id) = 1$: for factors of type II (see the article of Popa at `https://cutt.ly/KX8y0Fs`). This definition is natural in the sense that being a subsystem means being a tensor factor rather than subspace.

   The intuitive idea is that the density matrix for an infinite-D system identified as a unit operator gives as its trace total probability equal to one. These factors emerge naturally for free fermions. "Hyperfinite" expresses the fact that the approximation of a factor with its finite-D cutoff is an excellent approximation.

   HFFs are extremely flexible and can look like arbitrarily high-dimensional factor I_n. For instance, one can extract any matrix algebra $M^n(C)$ as a tensor factor so that one has $M = M^n(C) \otimes M^{1/n}$ by the multiplicativity of dimensions in the tensor product. Should one interpret this by saying that measurement can separate from a factor an $n-D$ Hilbert space and that $M^{1/n}$ is something that remains inaccessible to the measurements considered? If one introduces the notion of measurement resolution in this manner, the description of measurement could be based on factors of tyoe I_n.

3. The factors of type II_∞ are tensor products of infinite-D factors of type I and HFFs and could describe free bosons and fermions.

4. In quantum field theory (QFT), factors of type III appear and in this case the notion of trace becomes useless. These factors are pathological and in QFT they lead to divergence difficulties. The physical reason is the idea about point-like particles, which is too strong an idealization.

   In the TGD framework, the generalization of a point-like particle to 3-surface saves from these difficulties and leads to factors of type I and HFFs. In TGD, finite measurement resolution is realized in terms of a unique number theoretic discretization, which further simplifies the situation in the TGD framework.

2.2 Standard construction for the hierarchy of HFFs

Consider now the standard construction leading to a hierarchy of HFFs and their inclusions.

1. One starts from an inclusion $M \subset N$ of HFFs. I will later consider what these algebras could be in the TGD framework.

2. One introduces the spaces $L^2(M)$ *resp.* $L^2(N)$ of square integrable functions in M *resp.* N.

   From the physics point of view, bringing in "L^2" is something extremely non-trivial. Space is replaced with wave functions in space: this corresponds to what is done in wave mechanics, that is quantization! One quantizes in M, particles as points of M are replaced by wave functions in M, one might say.

3. At the next step one introduces the projection operator e as a projection from $L^2(N)$ to $L^2(M)$: this is like projecting wave functions in N to wave functions in M. I wish I could really understand the physical meaning of this. The induction procedure for second quantized spinor fields in H to the space-time surface by restriction is completely analogous to this procedure.

   After that one generates a HFF as an algebra generated by e and $L^2(N)$: call it $\langle L^2(N), e\rangle$. One has now 3 HFFs and their inclusions: $M_0 \equiv M$, $M_1 \equiv N$, and $\langle L^2(N), e\rangle \equiv M_2$.

   An interesting question is whether this process could generalize to the level of induced spinor fields?

4. Even this is not enough! One constructs $L^2(M_2) \equiv M_3$ including M_2. One can continue this indefinitely. Physically this means a repeated quantization.

One could ask whether one could build a hierarchy M_0, $L^2(M_0)$,..., $L^2(L^2...(M_0))..)$: why is this not done?

The hierarchy of projectors e_i to M_i defines what is called Temperley-Lieb algebra [A14] involving quantum phase $q = exp(i\pi/n)$ as a parameter. This algebra resembles that of S_∞ but differs from it in that one has projectors instead of group elements. For the braid group $e_i^2 = 1$ is replaced with a sum of terms proportional to e_i and unit matrix: mixture of projector and permutation is in question.

5. There is a fascinating connection in TGD and theory of consciousness. The construction of what I call infinite primes [K11, K6] is structurally like a repeated second quantization of a supersymmetric arithmetic quantum field theory involving fermions and bosons labelled by the primes of a given level I conjectured that it corresponds physically to quantum theory in the many-sheeted space-time.

   Also an interpretation in terms of a hierarchy of statements about statements about .... bringing in mind hierarchy of logics comes to mind. Cognition involves generation of reflective levels and this could have the quantization in the proposed sense as a quantum physical correlate.

2.3 Classification of inclusions of HFFs using extended ADE diagrams

Extended ADE Dynkin diagrams for ADE Lie groups, which correspond to finite subgroups of $SU(2)$ by McKay correspondence [A13, A12, A9], discussed from the TGD point of view in [L23], characterize inclusions of HFFs.

For a subset of ADE groups not containing E_7 and D_{2n+1}, there are inclusions, which correspond to Dynkin diagrams corresponding quantum groups. What is interesting that E_6 (tetrahedron) and E_8 (icosahedron/dodecahedron) appear in the TGD based model of bioharmony and genetic code but not E_7 (cube and octahedron) [L15].

1. Why finite subgroups of $SU(2)$ (or $SU(2)_q$) should characterize the inclusions in the tunnel hierarchies with the same value of the quantum dimension $M_{n+1} : M_n$ of quantum group?

   In the TGD interpretation M_{n+1} reduces to a tensor product of M_n and quantum group, when M_n represents reduced measurement resolution and quantum group the added degrees of freedom. Quantum groups would represent the reduced degrees of freedom. This has a number theoretical counterpart in terms of finite measurement resolution obtained when an extension of ... of rationals is reduced to a smaller extension. The braided relative Galois group would represent the new degrees of freedom.

2. One can algebraically identify HFF as a "tunnel" obtained by iterated standard construction as an infinite tensor power of $GL(2, c)$ or $GL(n, C)$. The analog of the McKay graph for the irreps of a closed subgroup of $GL(2, C)$ defines an invariant characterizing the fusion rules involved with the reduction of the Connes tensor products. This invariant reduces to the McKay graph for the tensor products of the canonical 2-D representation with the irreps of a *finite* rather than only closed subgroups of $SU(2)$. This must take place also for $GL(n, C)$. Why?

   The reduction of degrees of freedom implied by the Connes tensor product seems to imply a discretization at the level of $SU(2)$ and replace closed subgroups of $SU(2)$ with finite subgroups. This conforms with the similarity of the representation theories of discrete and closed groups. In the case of quantum group representations only a finite number of ordinary finite-D group representations survive.

   All this conforms with the TGD view about the equivalence of number-theoretic discretization and inclusions as descriptions of finite measurement resolution.

In the TGD framework, $SU(2)$ could correspond to a covering group of quaternionic automorphisms and number theoretic discretization (cognitive representations) would naturally lead to discrete and finite subgroups of $SU(2)$.

3 TGD and hyperfinite factors of type II_1: a bird's eye of view

In this section, a tentative identification of hyperfinite factors of type II_1 (HFFS) in the TGD framework [K12, K9] is discussed. Also some general related to the interpretation of HFFs and their possible resolution in the TGD framework are considered.

3.1 Identification of HFFs in the TGD framework

3.1.1 Inclusion hierarchies of extensions of rationals and of HFFs

I have enjoyed discussions with Baba Ilya Iyo Azza about von Neumann algebras. Hyperfinite factors of type II_1 (HFF) (`https://cutt.ly/lXp6MDB`) are the most interesting von Neumann algebras from the TGD point of view. One of the conjectures motivated by TGD based physics, is that the inclusion sequences of extensions of rationals defined by compositions of polynomials define inclusion sequences of hyperfinite factors. It seems that this conjecture might hold true!

Already von Neumann demonstrated that group algebras of groups G satisfying certain additional constraints give rise to von Neuman algebras. For finite groups they correspond to factors of type I in finite-D Hilbert spaces.

The group G must have an infinite number of elements and satisfy some additional conditions to give a HFF. First of all, all its conjugacy classes must have an infinite number of elements. Secondly, G must be amenable. This condition is not anymore algebraic. Braid groups define HFFs.

To see what is involved, let us start from the group algebra of a finite group G. It gives a finite-D Hilbert space, factor of type I.

1. Consider next the braid groups B_n, which are coverings of S_n. One can check from Wikipedia that the relations for the braid group B_n are obtained as a covering group of S_n by giving up the condition that the permutations σ_i of nearby elements e_i, e_{i+1} are idempotent. Could the corresponding braid group algebra define HFF?

   It is. The number of conjugacy classes $g_i\sigma_i g_i^{-1}$, $g_i == \sigma_{i+1}$ is infinite. If one poses the additional condition $\sigma_i^2 = U \times 1$, U a root of unity, the number is finite. Amenability is too technical a property for me but from Wikipedia one learns that all group algebras, also those of the braid group, are hyperfinite factors of type II_1 (HFFs).

2. Any finite group is a subgroup G of some S_n. Could one obtain the braid group of G and corresponding group algebra as a sub-algebra of group algebra of B_n, which is HFF. This looks plausible.

3. Could the inclusion for HFFs correspond to an inclusion for braid variants of corresponding finite group algebras? Or should some additional conditions be satisfied? What the conditions could be?

Here the number theoretic view of TGD could comes to the rescue.

1. In the TGD framework, I am primarily interested in Galois groups. The vision/conjecture is that the inclusion hierarchies of extensions of rationals correspond to the inclusion hierarchies for hyperfinite factors. The hierarchies of extensions of rationals defined by the hierarchies of composite polynomials $P_n \circ ... \circ P_1$ have Galois groups, which define a hierarchy of relative Galois groups such that the Galois group G_k is a normal subgroup of G_{k+1}. One can say that the Galois group G is a semidirect product of the relative Galois groups.

2. One can decompose any finite subgroup to a maximal number of normal subgroups, which are simple and therefore do not have a further decomposition. They are primes in the category of groups.

3. Could the prime HFFs correspond to the braid group algebras of simple finite groups acting as Galois groups? Therefore prime groups would map to prime HFFs and the inclusion hierarchies of Galois groups induced by composite polynomials would define inclusion hierarchies of HFFs just as speculated.

One would have a deep connection between number theory and HFFs.

3.1.2 How could HFFs emerge in TGD?

What could HFFs correspond to in the TGD framework? Consider first the situation at the level of M^8.

1. Braid group $B(G)$ of group (say Galois group as subgroup of S_n) and its group algebra would correspond to $B(G)$ and $L^2(B(G)$). Braided Galois group and its group algebra could correspond to $B(G)$ and $L^2(B(G))$.

   The inclusion of Galois group algebra of extension to its extension could naturally define a Connes tensor product. The additional degrees of freedom brought in by extension of extension would be below measurement resolution.

2. Composite polynomials $P_n \circ .... \circ P_1$ are used instead of a product of polynomials naturally characterizing free n-particle states. Composition would describe interaction physically: the degree is the product of degrees of factors for a composite polynomial and sum for the product of polynomials.

   The multiplication rule for the dimensions holds also for the tensor product so that functional composition could be also seen as a number theoretic correlate for the formation of interacting many particle states.

3. Compositeness implies correlations and formation of bound states so that the number of degrees of freedom is reduced. The interpretation as bound state entanglement is suggestive. This hierarchical entanglement could be assigned with directed attention in the TGD inspired theory of consciousness [L13].

   An alternative interpretation is in terms of braids of braids of ... of braids with braid strands at a given level characterized by the roots of P_i. These interpretations could be actually consistent with each other.

4. Composite polynomials define hierarchies of Galois groups such that the included Galois group is a normal subgroup. This kind of hierarchy could define an increasing sequence of inclusions of braided Galois groups.

Consider the situation at the H level.

1. At the level of H, elements of the algebras $A \in \{SSA, Aff, I\}$a associated with super-symplectic symmetries acting at δM^4_+, affine isometries acting at light-like partonic orbites, isometries of δM^4_+, are labelled by conformal weights coming as non-negative integers. Also algebraic integers can be considered but for physical states conformal confinement requires integer valued conformal weights.

2. One can construct a hierarchy of representations of A such that subalgebras A_n with conformal weights $h \geq 0$ coming as multiples of n and the commutator $[A_n, A]$ annihilate the physical states. These representations are analogous to quantum groups and one can say that A_n defines a finite measurement resolution in A. A_{nk}, $k \geq 1$ is included by A_n for and one has a reversed sequence of inclusions.

   One can construct inclusion hierarchies defined by the sequences $1 \div n_1 \div n_2 \div ....$ "$n_{-1} = 1$" corresponds to SSA. The factor spaces $A_{n_k}/A_{n_{k+1}}$ are analogs of quantum group-like objects associated with Jones inclusions and the interpretation is in terms of finite measurement resolution defined by $A_{n_{k+1}}$.

   The factor spaces A/A_{n_k} define inclusion hierarchies with an increasing measurement resolution.

3.2 Some objections against HFFs

One cannot avoid philosophical considerations related to the notion of probability and to the interpretations of quantum measurement theory (https://cutt.ly/YXxSLS1).

3.2.1 Standard measurement theory and HFFs

The standard interpretations of quantum measurement theory are known to lead to problems in the case of HFFs.

1. An important aspect related to the probabilistic interpretation is that physical states are characterized by a density matrix so that quantum theory reduces to a purely statistical theory. Therefore the phenomenon of interference central in the wave mechanics does not have a direct description.

   Another problem is that for HFFs, pure states do not exist as so-called normal states, which are such that it is possible to assign a density operator to them. This is easy to understand intuitively since by the $Tr(Id) = 1$ property of the unit matrix, there is no minimal projection. Selection of a ray would correspond to an infinite precision and delta function type density operator. The axiom of choices in mathematics is quite a precise analogy.

   One can of course argue that even if pure states as normal states are possible, in practice the studied system is entangled with the environment and that this forces the description in terms of a density matrix even when pure states are realized at the fundamental level.

2. In the purely statistical approach, the notion of quantum measurement must be formulated in terms of what occurs for the density matrix in quantum measurement. The expectation value of any observable A for the new density matrix generated in the measurement of observable O with a discrete spectrum must be a weighted sum for the expectations for the eigenstates of the observable with weights given by the state function reduction probabilities.

   Problems are however encountered when the spectrum contains discrete parts. In the TGD framework, the number theoretic discretization would make it possible to avoid these problems.

3.2.2 Should density matrix be replaced with a more quantal object?

These problems force us to ask whether there could be something deeply wrong with the notion of density matrix? The TGD inspired view of HFFs [K12, K9] suggests a generalization of the state as a density matrix to a "complex square root" of the density matrix. At the level of WCW as vacuum functional, it would be proportional to exponent of a real valued Kähler function of WCW identified as Kähler action for the space-time region as a preferred extremal and a phase factor defined by the analog of of action exponential. Zero energy state would be proportional to an exponent of Kähler function of WCW identified as Kähler action for space-time surface as a preferred extrema.

3.2.3 Problems with the interpretations of quantum theory

HFFs based probability concept has also problems with the interpretations of quantum theory, which actually strongly suggest that something is badly wrong with the standard ontology.

1. In TGD, this requires a generalization of quantum measurement theory [L8] [K13] based on zero energy ontology (ZEO) and Negentropy Maximization Principle (NMP) [K4] [L3], which is consistent with the second law [L16]. What is essential is that physics is extended to what I call adelic physics [L5, L6] to describe also the correlates of cognition. This brings in a measure for conscious information based on a p-adic generalization of Shannon entropy.

2. ZEO [K13] is forced by an almost exact holography in turn implied by general coordinate invariance for space-time as 4-surface. States in ZEO are superpositions of classical time evolutions and and is replaced by a new one in a state function reduction (SFR) [L8, L22]. The determinism of the unitary time evolution is consistent with the non-determinism of SFR. The basic problem of quantum measurement theory disappears since there are two times and two causalities. Causality of field equations and geometric time of physicists can be assigned to the classical time evolutions. The causality of free will and flow of experienced time can be assigned to a sequence of SFRs. The findings of Minev et al [L7] provide support for ZEO [L7].

Quantum measurement as a reduction of entanglement can in principle occur for any entangled system pair if NMP favors it. There is no need to assume mysterious decoherence as a separate postulate. By NMP, entanglement negentropy can also increase by the formation of entangled states. Since entanglement negentropy is the sum of positive p-adic contribution and negative contribution from real entanglement and is positive, the increase of negentropy is consistent with the increase of real entanglement entropy.

However, since classical determinism is slightly broken [L19] (there is analogy with the non-uniqueness of the minimal surfaces spanned by frames), the holography is not quite exact. This has important implications for the understanding of the space-time correlates of cognition and intentionality in the TGD framework.

3.2.4 The notion of finite measurement resolution and probabilistic interpretation

One can also ask whether something could go wrong with the quantum measurement theory itself. This notion of quantum measurement does not take into account the fact that the measurement resolution is finite.

The notion of finite measurement resolution realized in terms of inclusion, replacing Hilbert space ray with the included factor and reducing state space to quantum group like object, could allow us to overcome the problems due to the absence of minimal projectors for HFFs implying that the notion of Hilbert space ray does not make sense.

Quantum group like object would represent the degrees of freedom modulo finite measurement resolution described by the included factor. The quantum group representations form a finite subset of corresponding group representations and the state function reductions could occur to quantum group representations and the standard quantum measurement theory for factors of type I would generalize.

3.2.5 Connes tensor product and finite measurement resolution

In the TGD framework Connes tensor product could provide a description of finite measurement resolution in terms of inclusion.

1. In the TGD framework, inclusion of HFFs are interpreted in terms of measurement resolution. The included factor $M \subset N$ would represent the degrees of freedom below measurement resolution. N as M module would mean that M degrees of freedom are absorbed to the coefficient ring and are not visible in the physical states. Complex numbers as a coefficient ring of the Hilbert space are effectively replaced with M. In the number theoretic description of the measurement resolution, the extension of extension is replaced with the extension. The quantum group, N as M, quantum group with quantum dimension $N : M$ would characterize the observable degrees of freedom.

   This fits with the hierarchy of SSA_n:s. SSA_{n+1} would take the role of M and SSA_n that of N. This conforms with the physical intuition. Since n corresponds to conformal weight, the large values of n would naturally correspond to degrees of freedom below UV cutoff.

   Could also IR cutoff have a description in the super symplectic hierarchy of SSA_n:s. It should correspond to a minimal value for conformal weight. The finite size of CD defining a momentum unit gives a natural IR cutoff. The proposal is that the total momentum assignable to the either half-cone of CD defines by $M^8 - H$ duality the size scale L as $L = h_{eff}/M$ [L11, L12].

2. For the hierarchies of extensions of rationals the upper levels of the extension hierarchy would not be observed. The larger the value of $n = h_{eff}/h_0$, n a dimension of extension of rationals associated with polynomial P defining the space-time region by $M^8 - H$ duality, the longer the quantum coherence scale.

   In this case large values for the dimension of extension would correspond to IR cutoff. Therefore UV and IR cutoffs would correspond to number theoretic and geometric cutoffs. This conforms with the view that $M^8 - H$ duality as an analog of Langlands duality is between number theoretic and geometric descriptions.

3. Duality suggests that also UV cutoff should have a number theoretic description. In the number theoretic situation, Galois confinement for these levels might imply that they are indeed unobservable, just like color-confined quarks. In fact, the hypothesis $n = h_{eff}/h_0$, n a dimension of extension of rationals associated with polynomial P defining the space-time region by $M^8 - H$ duality, for the effective Planck constant leads to estimate for ordinary Planck constant as $h = n_0 h_0$ where n_0 corresponds to the order of permutation group S_7.

   Could the interpretation be that these degrees of freedom are Galois confined and unobservable in the scales at which measurements are performed. Smaller values of h_{eff} would appear only in length scales much below the electroweak scale and at the limit of CP_2 scale?

3.2.6 How finite measurement resolution could be realized using inclusions of HFFs?

The basic ideas are that finite measurement resolution corresponds to inclusions of HFFs on one hand, and to number theoretic discretizations defined by extensions of rationals. In both cases one has inclusion hierarchies.

One can consider realizations at the level of WCW (geometry) and at the level of number theory in terms of adelic structures assignable to the extensions of rationals. Space-time surfaces can be discretized and this induces discretization of WCW. Even more, WCW should be in some natural manner effectively discrete.

In [K2, K1, K7] the construction WCW Kähler metric is considered and the mere existence of the K "ahler metric is expected to require infinite-D isometry group and imply constant curvature property. The Kähler function K is defined in terms of action consisting of the Kähler action and volume part for a preferred extremal (PE). There are however zero modes present and the metric depends on the zero modes. Twistor lift fixes the choices of H uniquely [L20, L21].

How to define WCW functional integral and how to discretize it? I have proposed that the Gaussian approximation to WCW integration is exact and allows to define a discretization in terms of the maxima (maybe also other extrema) of Kähler function. The proposal is that the exponential of Kähler function should correspond to a number theoretic invariant, perhaps the discriminant of the polynomial P defining Pe by $M^8 - H$ duality.

Consider first the standard realization of the restriction $P : N \to M$ reducing the measurement resolution.

1. The definition of a unitary S-matrix for HFFs is non-trivial. Usually one considers only density matrices expressible in terms of projection operators P to subspaces of HFF.

   I have earlier proposed the notion of a complex square root of the density matrix as a generalization of the density matrix. In a direct sum representation of S over projections, in which S-matrix is diagonal, and the projection operators would be multiplied by phase factors. This definition looks sensible at the level of WCW but perhaps as a generalization of the density matrix rather than the S-matrix.

   The exponent of Kähler function could have a modulus multiplied by a phase factor. Also an additional state dependent phase factor can be considered. The mathematical existence of the WCW integral fixes the modulus essentially uniquely to an exponent of Kähler function K multiplied by the metric volume element. K could also have an imaginary part.

2. The projected S-matrix PSP is unitary if the projection operator P must commute with S. S-matrix is realized at the level of HFFs so that the matrix representation does not make sense in a strict sense since the notion of ray is not sensical.

3. Projection $N \to M$ respects unitarity only if P commute with S and $S^{\dagger}$. The S-matrix does not have matrix elements between M and N. This is a very tough condition.

How the finite measurement resolution could be realized in the TGD framework?

1. In WCW spin degrees of freedom plus algebras A_n. Number theoretic degrees of freedom are discrete and correspond to various p-adic degrees of freedom. Continuous WCW is associated with the real part of the adelic structure. Its number theoretic parts correspond to the p-adic degrees of freedom, which are discrete.

2. Discretization could be a natural and necessary part of the definition of WCW. Could discrete WCW degrees of freedom be identified in terms of symplectic and number theoretic invariants? They would represent for WCW spinor fields scalar degrees scalar degrees of freedom separable from spin degrees of freedom representable in terms of algebras A_n. These two kinds of degrees of freedom correspond to M and M' if the proposed general picture is correct.

Measurement resolution would be realized in terms of braid group algebras and algebras A_n defining the measurement resolution. What does this mean at the level of WCW?

1. Bosonic generators of SSA_n and possible other algebras A_n define tangent space basis for WCW. The gauge conditions stating that A_n and $[A_n, A]$ annihilate WCW spinor fields define a finite measurement resolution selecting only a subset of tangent space-generators and their super counterparts.

2. Consider first ideal measurement resolution in a function space. There is a complete basis of scalar functions Φ_m in a given space. The sum $\overline{\Phi}_m(x)\Phi_m(y) = \delta(x,y)$ would hold true for an infinite measurement resolution.

   In a finite measurement resolution one uses only a finite subset of the scalar function basis, and completeness relation becomes non-local and is smoothed out: $\delta(x,y) \rightarrow D(x,y)$, which is non-vanishing for different point pairs x, y.

3. The condition of finite measurement resolution should define a partition of WCW to disjoint sets. In real topology, the condition $|x-y|^2$ would define a natural measurement resolution but would not define a partition.

   In p-adic topology, the situation is different: the p-adic distance function $d(x-y)$ has values p^{-n} and the sets $d(x-y) < d$ are either disjoint or identical. One would have the desired partition. Therefore it seems that p-adicization is essential and the p-adic variants of WCW, or rather regions of WCW, obtained by discretization could allow partitions corresponding to various p-adic number fields forming the adele. Different p-adic representations of algebras A_n would define measurement resolutions.

   There is a connection with spin glasses where spin energy landscape consisting of free energy minima allows ultrametric topology: p-adic topologies are indeed ultrametric. The TGD view of spin glasses is discussed in [L17]. One expects the decomposition of WCW to different p-adic topologies with ramified primes of polynomial P defining the p-adic sectors to which a given space-time surface can belong.

4. The consistency condition is that the transition probabilities $P(m \rightarrow n)$ between the states satisfying the gauge conditions representing finite measurement resolution, predicted by S-matrix or its TGD counterpat, should be constant should be constant in the subsets of WCW for which the completeness relation gives a non-vanishing $D(x,y)$ for the point pairs (x,y).

5. Does WCW have hierarchies of partitions such that the constancy of $P(m \rightarrow n)$ holds true within each partition?

   Do these partitions correspond to hierarchies of inclusions of HFFs defining increasing resolution? $M^8 - H$ duality does not allow all kinds of hierarchies. The hierarchies should be induced by the hierarchies of extensions of rationals. As the measurement precision increases, the partition involves an increasing number of sets and at the limit of ideal measurement resolution, the partition consists of algebraic points of WCW and of space-time surfaces.

6. $P = Q$ condition implying that space-time surfaces correspond to infinite prime, could appear as a consistency condition for allowed hierarchies. Preferred extremals and preferred polynomials would correspond to each other. Note that $P = Q$ conditions fixes the scaling of P.

In the TGD framework, one can challenge the idea, originally due to Wheeler, that transition probabilities are given by a unitary S-matrix.

1. The TGD based proposal is that in spin degrees of freedom, that is for many-fermion states for a given space-time surface, the counterpart of S-matrix could be be given by the analog of Kähler metric in the fermionic Hilbert space [L14]. This would mean a geometrization of quantum theory, at least in fermionic degrees of freedom.

   The transition probabilities would be given by $P(m \to n) = K_{\overline{m}n}K^{\overline{n}m}$ and the properties of Kähler metric K give analogs of unitary conditions and probability conservation plus some prediction distinguishing the proposal from the standard view.

2. In the infinite-D situation, the existence of Hilbert space Kähler metric in the fermionic sector is an extremely powerful condition and one expects that the Kähler metric is a unique constant curvature metric allowing a maximal group of isometries. This, together with p-adization, would help to satisfy the constancy conditions for $P(m \to n)$ inside the sets for which $D(x, y)$ is non-vanishing. In fact, one expects that since super-generators are proportional to isometry generators contracted with WCW gamma matrices the metric in the fermionic degrees of freedom is induced by Kähler metric in the basis of isometry generators.

3. This condition could allow a generalization to include the states obtained by application of the bosonic generations of A_n the to ground state. This would mean that in bosonic degrees of freedom Kähler metric of WCW in the isometry basis defines the transition probabilities. Tangent vectors of WCW correspond to the isometry generators. An arbitrary number of isometry generators is involved in the definition of the state. However, the Kähler metric of WCW induces a Kähler metric in the algebra generated by the isometry generators, which is analogous to the algebra of tensors.

4 $M^8 - H$ duality and HFFs

$M^8 - H$ duality [L11, L12] gives strong constraints on the interpretation of HFFs at the number theoretic M^8 side and the geometric H side of the duality. One must also understand the relation between $M^8 - H$ duality and $M - M'$ duality, identifiable as quantum-classical correspondence (QCC).

Although McKay correspondence [A13, A12, A9, A7, A6] is not quite at the core of $M^8 - H$ duality, it is difficult to avoid its discussion. I have considered McKay correspondence also before [L4, L9, L10, L23].

4.1 Number theoretical level: M^8 picture

4.1.1 Braided Galois group algebras

For n-braids the permutation group has extension to a braid group B_n defining an infinite covering of S_n for which permutation corresponds to a geometric operation exchanging the two strands of a braid. There are also hierarchies of finite coverings.

S_n is replaced with the Galois group which is a subgroup of S_n and the property of being a subgroup of S_n allows to identify a braided Galois group as a braided Galois subgroup of braided S_n. In the same way one can identify the braided Galois group algebra defining HFF as a sub-algebra of HFF associated with braid group algebra defined by S_n. One can ask whether the property of being a number theoretic braid could be interpreted as a kind of symmetry breaking to S_n to the Galois group of P.

$M^8 - H$ duality [L11, L12] suggests that the roots correspond to braid strands of geometric braids in H. If so, the braided Galois group would be both topological and number theoretic: topology, natural at the level of H, and number theory, natural at the level of M^8, would meet by $M^8 - H$ duality.

This picture looks nice but one can make critical questions.

1. Can the n roots really correspond to n braid strands at the level of H? The n roots correspond to, in general complex, algebraic numbers associated with the extension of rationals. The real projections correspond to mass shells with different mass values mapped to light-cone proper time surfaces in H by $M^8 - H$ duality. Therefore the action of the Galois group changes

mass squared values and does not commute with Lorentz transformations. This suggests a violation of causality.

Should one restrict the Galois group to the isotropy group of a given root? This would mean number theoretic symmetry breaking and could relate to massivation. This restriction would however trivialize the braid.

2. Zero energy ontology (ZEO) could come to the rescue here. In fact, ZEO implies space-time surfaces are the basic objects rather than 3-surfaces so that quantum states are superpositions of space-time surfaces as preferred extremals (PEs). This is forced by the slight violation of determinism of field equations implying also a slight violation of ideal holography.

   Space-time surfaces are minimal surfaces [L19] analogous to soap films spanned by frames and there can be a slight violation of the strict determinism localized to frames as already 2-D case suggests. This could be also seen as violation of classical causality. At the level of consciousness theory it would be a classical correlate for the non-determinism of intentional free will.

   In particular, time-like braids for which the braiding is time-like and corresponds to a dynamical dance pattern, make sense. For these braids one can in principle select the mass squared value mapped to a value of light-cone proper time a to belong to the braid. The values of a need not be the same.

Also Galois confinement, which is a key aspect of the number theoretic vision, is involved.

1. Galois confinement states that physical states transform trivially under the Galois group of extension. This condition for physical states follows as a consequence of periodic boundary conditions for causal diamond (CD), which takes the role of box for a particle in a box.

   A weaker condition would be that singlet property holds only for the isotropy group of a given root of the polynomial P characterizing the space-time region and corresponding to mass squared value and at the level of H to a value of the light-cone proper time a.

2. In M^8, the momenta of particles are points at the mass shells of $M^4 \subset M^8$ identifiable as hyperbolic spaces $H^3 \subset M^4$ defined with mass squared values defined as the roots of P. The momenta correspond to algebraic integers (the momentum unit is defined by CD) for the extension defined by P, and in general they are complex. The interpretation is as virtual particles which form physical particles as composites. The physical states must have total momenta, which are ordinary integers. This gives the simplest form of Galois confinement.

3. Commutativity with the Lorentz group would favor the isotropy group instead of the full Galois group. One must be however very cautious since in zero energy ontology (ZEO) physical states correspond to a superposition of space-time surfaces and time-like braids are natural. There is a small violation of strict determinism at the level of preferred extremas. The labelling of braid strands based on the images of roots as mass squared values at level of H is quite natural and is not in conflict with causality.

The Galois group for a polynomial $P_n \circ ... \circ P_1$ has a decomposition to normal subgroups GA_i acting as Galois groups for the i:th sub-extension.

1. The number of roots is a product of the numbers of roots for P_i. Therefore the natural identification is that number theoretic braid groups allow a natural interpretation in terms of braids of braids ... of braids.

2. This hierarchy defines an inclusion hierarchy for the braided HFFs assignable to the polynomials $P_k \circ ... \circ P_1$, $k = 1, ..., n$. It is not quite clear to me whether these inclusions reduce to Jones inclusions and whether one can characterize the inclusions in the sequence by the same invariants as in the case of Jones inclusions.

3. In this picture the Connes tensor product would correspond to formation of composite polynomials $P \circ Q$. The reduction in the number of degrees of freedom from that for the ordinary tensor product of braided Galois group algebras would be due to interactions described in terms of polynomial decomposition. Various braids in the hierarchy could correspond to braids at different sheets of the many-sheeted space-time.

4. Any normal subgroup Gal_i of Galois group Gal defining a sequence of inclusions of normal sub-groups Gal_i can be trivially represented. By normal subgroup property, the elements of Gal can be represented as semidirect products of elements of the factor groups $G_i = Gal_i/Gal_{i-1}$. Any representation of Gal can be decomposed to a direct sum of tensor products of representations of G_i.

   From this decomposition it is clear that any group G_i in the decomposition can be trivially represented so that one obtains a rich structure of representation in which some G_i:s are trivially represented.

4.1.2 How could the degrees of prime polynomials associated with simple Galois groups and ramified primes relate to the symmetry algebras acting in H?

The goal is to relate various parameters characterizing polynomials P for which braided Galois group algebras define HFFs to the parameters labelling the symmetry algebras defining hierarchies of HFFs at the level H. There are good reasons to believe that polynomial composition defines inclusion of HFFs and that this inclusion induces the inclusions for the symmetry algebras A_n at the level of H.

One can identify simple Galois groups as prime groups having no normal subgroups. The polynomial P associated with a simple Galois group cannot have no non-trivial functional decomposition $P_n \circ ... \circ P_1$ if one stays in the field of rationals (say). This leads to the notion of prime polynomials. Note that this notion of primeness does not correspond to the irreducibility stating that polynomials with coefficients in a given number field do not allow decomposition to lower degree polynomials.

A polynomial P is also partially characterized by ramified primes and discriminant defines a Galois invariant for the polynomial as also the symmetric polynomials formed from the roots.

How do these two notions of primeness relate to the p-adic prime decomposition of adelic structures defined by the algebras A_n, which act at the level of H and decomposed adelically to a tensor product of all $A_{n,p}$:s?

Simple Galois groups correspond to prime polynomials. This notion looks fundamental concerning the understanding of the situation at the level of H.

1. Polynomials can be factorized into composites of prime polynomials [A3, A11] (`https://cutt.ly/HXAKDzT` and `https://cutt.ly/5XAKCe2`). A polynomial, which does not have a functional composition to lower degree polynomials, is called a prime polynomial. It is not possible to assign to prime polynomials prime degrees except in special cases. Simple Galois groups with no normal subgroups must correspond to prime polynomials.

2. For a non-prime polynomial, the number N of the factors P_i, their degrees n_i are fixed and only their order can vary so that n_i and $n = \prod n_i$ is an invariant of a prime polynomial and of simple Galois group [A3, A11]. Note that this composition need not exist for monic polynomials even if the Galois group is not simple so that polynomial primes in the monic sense need not correspond to simple Galois groups.

3. The number of the roots of P_i is given by its order n_i, and since Galois group and its braided variant permute the roots as subgroup of S_{n_i}, it is natural to assume that the roots define an n_i-braid. The composite polynomial would define braid of braids of ... of braids. At the level of H the braid strands would correspond to flux tubes and braiding would have a geometric interpretation.

4. The integer n characterizing the algebra A_n acting in H would naturally correspond to the the degree of n of P and the decomposition of P to polynomial primes would naturally correspond to an inclusion hierarchy A_{n_i} $A_{n_1} \subset A_{n_1 n_2} \subset ... \subset A_n$ with improving resolution allowing to see braids and braids of braids.

   The corresponding factor spaces realizing the notion of finite measurement resolution, would be analogous to quantum groups obtained when some number of the highest levels in the hierarchy of braids in the braid of braids of ... braids are neglected and the entire algebra is replaced with a quantum group-like structure. This means cutting off some number of the highest levels in the tree-like hierarchy. The trunk is described by a quantum group-like object.

5. This hierarchy corresponds to the hierarchy of Galois groups as normal subgroups assignable to braids in the decomposition and the hierarchy of corresponding braided Galois group algebras defining inclusions of HFFs. Galois group algebras would act as braid groups inc corresponding algebras A_n. Therefore number theoretic and geometric views would fuse together.

6. Connes tensor product is a central notion in the theory of HFFs and it could be naturally associated with the inclusions of brided Galois group algebras. The counterpart for the quantum group as factor space N/M of the factors would correspond to the inclusion $Gal_{i-1} \subset Gal_i$ as a normal subgroup. The inclusion defines group $G_i = Gal_i/Gl_{i-1}$. Also its braided variant is defined. The factor space of braided group algebras would be the counterpart of the quantum group G_i.

   Note that these quantum group-like objects could be much more general than the quantum groups defined by subgroups of $SU(2)$ appearing in Jones inclusions.

What about the interpretation of the ramified primes, which are Galois invariants as also the root spectrum (but not the roots themselves) and depends on the polynomial.

In accordance with the proposed physical interpretation of the ramified primes as preferred p-adic primes labelling particles in p-adic thermodynamics, ramified primes p_i would define preferred p-adic primes for the p-adic variants of the algebras A_n in the adelic generalization of A_n as tensor product of p-adic representations of $A_{n,p}$ of A_n. A_{n,p_i} would be physically and also mathematically special.

Both the degree n as the number of braids of P and the ramified primes of P would dictate the physically especially relevant algebras A_{n,p_i}. For instance, un-ramified primes could be such that corresponding p-adic degrees of freedom are not excited.

4.2 Geometric level: H picture

4.2.1 The hierarchies of algebras SSA_n, Aff_n and I_n

The algebras $A_n \in \{SSA_n, Aff_n, I_n\}$ for $n = p$ acting at the level of WCW seem to have special properties since the values of the conformal weights for the the factor algebras defined by the conditions that A_n and $[A_n, A]$ annihilate physical states, allow the structure of finite field $G(p)$ or even its extension $G(p, k)$ for conformal weights in extension of rationals. The representations would be finite-D. Also the values $n = p^k$ seem special and the finite field representations of SSA_p could be extended to p-adic representations.

This raises the question, whether one could regard n as a p-adic number? The interpretation of n as the number of braid strands assignable to roots of the polynomial P with degree n defining the space-time surface, looks more approriate since it allows braid group algebra of P to act in SSA_n, This identification does not favor this interpretation.

A more plausible interpretation is that the p-adic primes, identifiable as ramified primes of P, characterize the p-adic representations of SSA_n. This also conforms with the interpretation of preferred p-adic primes characterizing elementary particles as ramified primes.

The polynomials with prime degree could be however physically special. The algebras SSA_p, with p defining the degree of polynomial p allow finite field representations, which extend to p-adic representations and one can ask whether the prime decomposition of n could allow some kind of inclusion hierarchy of representations.

This would also give a possible content for the p-adic length scale hypothesis $p \simeq 2^k$, k prime, or its generalization involving primes near powers of prime $q = 2, 3, 5, ...$ A more general form of p-adic length scale hypothesis would be $p \simeq q^n$, n the degree of P.

4.2.2 Commutants for algebras A_n and braid group algebras

For the super $A \in \{SSA, Aff, I\}$, the inclusion An_{nk} to SSA_n should define a Connes tensor product. One would obtain inclusion hierarchies labelled by divisibility hierarchies $n_1 \div n_2 \div ....$ For braid group algebras one obtains similar hierarchies realized in terms of composite polynomials.

What about the already mentioned "classical" degrees of freedom associated with the fluxes of the induced Kähler form? They should be included to M' at the level of H. The hierarchies of flux tubes within ... within flux tubes correspond to the hierarchies assignable to M' at the level of H.

The number theoretic degrees of freedom identifiable as invariants of Galois groups should be included to M' at the number theoretical level. The hierarchies of roots assignable to composite polynomials $P_n \circ ... \circ P_1$ with roots assigned to the strands of time like braid strands could correspond to these hierarchies at the level of M^8.

4.3 Wild speculations about McKay correspondence

McKay correspondence is loosely related to the HFFs in TGD framework [L4, L10, L9, L23] and I cannot avoid the temptation to try to understand it in TGD framework.

1. The origin of the McKay graphs for inclusions is intuitively understood. Representations of finite subgroups of $SU(2)$ are assignable to 2-D factors. These representations could correspond to closed subgroups of quaternionic $SU(2)$ on the basis of the reduction to $M_2(C) \otimes M_2(C) \otimes$ ..... A reduction of degrees of freedom happens for HFFs since they are subalgebras of $B(H)$ and this could reduce the closed subgroup to a finite subgroup.

   Also the interpretation N as tensor product of M and quantum group $SU(2)$ suggests the same since quantum groups have a finite number of irreps, when q equal is a root of unity. The analog of McKay graph coding fusion rules for the quantum group tensor products would reduce to McKay graphs.

2. Why would the McKay graphs for finite subgroups of $U(2)$ correspond to those for affine or ordinary Lie algebras? Could these Lie-algebras emerge from the inclusions. This is a mystery, at least to me.

3. In the TGD framework one can ask why there should be Weyl group of extended ADE Dynkin diagram assignable to SSA_n? SSA_n defines a representation of SSA with SSA_n and $[SSA_n, SSA]$ acting trivially. Could this representation correspond to an affine or ordinary ADE algebra? Similar question makes sense for all algebras $A_n \in \{SSA_n, Aff_n, I_n\}$. A_nn would define a cutoff of the SSA so that all generators with conformal weight larger than n would be represented trivially.

   Note that for $n = p$, the conformal weights of A_n would define a finite field and if algebraic integers also its extension. This case could correspond to polynomials defining cyclic extension of order p with roots coming as roots of unity.

4. The Weyl groups assignable to the "factor algebra" of SSA_n defined by the gauge conditions for A_n and $[A_n, A]$ and proposed to reduce to ADE type affine or ordinary Lie algebra should relate to Galois groups for polynomial P with degree n as number of braid strands.

   (a) Could the braid strands correspond to the roots of ADE algebra so that roots in the number theoretic sense would correspond to the roots in the group theoretic sense? This would conform with Langlands correspondence [A1, A5, A4] discussed from the TGD perspective in [K10] [L1, L2].

   (b) Could the Weyl groups allow identification as subgroups of corresponding Galois groups?

   Note that simple Galois groups correspond to so-called prime polynomials [A3, A11] allowing no decomposition to polynomials of lower degree so that the preferred values of n would correspond to prime polynomials.

5. Affine electroweak and color algebras an their M^4 counterparts would be special since they wuuld not emerge a dynamical symmetries of SSA_n but define algebras Aff_n and I_n related to the light-like partonic orbits. They would also correspond to symmetries both at the level of M^8 and H.

This inspires the following questions, which of course look very naive from the point of view of a professional mathematician. My only excuse is the strong conviction that the proposed picture is on the right track. I might be wrong.

1. The Jones inclusion of HFFs [A8, A15, A16] involves an extended or ordinary ADE Dynkin diagram assignable also to finite subgroups of $SU(2)$ by McKay correspondence [A13].

   Could the Weyl group of an extended ADE diagram really correspond to an affine algebra or quantum group assignable to A_n? If so, one would have dynamical symmetries and should relate to the "factor" space SSA/SSA_n in which SSA_n defines a measurement resolution.

2. HFF can be regarded algebraically as an infinite tensor power of $M_2(C)$. Does the representation as a 2×2 matrix imply the emergence of representations of a closed subgroup of $SU(2)$ or its quantum counterpart. Could the reduction of degrees of freedom due to the finite measurement resolution imply that the closed subgroup reduces to a finite subgroup?

3. The algebraic decomposition of HFF to an infinite tensor power of $M^2(C)$ would suggest that the including factor N with dimension 1 is equal to $M^{d_q}\otimes M^{1/d_q}$, where d_q is the quantum dimension characterizing eith M or N. Could these two objects correspond to an ADE type affine algebra and quantum group with inverse quantum dimensions? Or could either of them correspond to ADE type affine algebra or quantum group?

4. Could one think that the analog of McKay graph for the quantum group-like object assignable to affine group by a finite measurement resolution reduces to the McKay graph for a finite subgroup of $SU(2)$ because only a finite number of representations survives?

5. Could the finite subgroups of $SU(2)$ correspond to finite subgroups for the covering group of quaternion automorphisms acting naturally in M^8? Could these finite subgroups correspond to finite subgroups of the rotation group $SU(2)$ at H side?

Could only the n_C (dimension of Cartan algebra) roots appearing in the Dynkin diagram be represented as roots of a polynomial P in extension of rationals or its quantum variant? This option fails since the Dynkin diagram does not allow a symmetry group identifiable as the Galois group. The so called Steinberg symmetry groups (`https://cutt.ly/GXMb8Si`) act as automorphisms of Dynkin diagrams of ADE type groups and seem quite too small and fail to be transitive as action of the Galois group of an irreducible polynomial is.

M^8-H duality inspires the question whether a subgroup of Galois group could act as the Weyl group of ADE type affine or ordinary Lie algebra at H side.

1. The Galois group acts as a braid group and permutes the roots of P represented as braid strands. Weyl group permutes the roots of Lie algebra

   The crazy question is whether the roots of P and roots of the ADE type Lie-algebra could correspond to each other. Could the roots of P in $N\rightarrow 1$-correspondence with the non-vanishing roots of the representation of Lie algebra or of its affine counterpart containing an additional root corresponding to the central extension?

   If the roots appearing in the Dynkin diagram correspond to a subset of roots of polynomial P, the Weyl group could correspond to a minimal subgroup of the Galois group generated by reflections and generating all non-vanishing roots of the Lie algebra.

2. The action of the Weyl group should give all roots for the representation of G. Could the Weyl group, which is generated by reflections, correspond to a minimal subgroup of Gal giving all roots as roots of P when applied to the McKay graph?

   The obvious objection is that the order of the Weyl group increases rapidly with the order of the Cartan group so that also the Gal and also the order of corresponding polynomials P would increase very rapidly. Gal is a subgroup of S_n having order $n!$ for a polynomial of degree n so that the degree of P need not be large and this is what matters.

   If the m braid strands labelled by the m roots correspond to the roots of the affine algebra, it would be natural to assign affine algebra generators to these roots with the braid strands.

The condition $n = Nm$ implies that m divides n. For $Gal = S_n$ with order $n!$ this condition is very mild. $Gal = Z_p$ fixes the Lie algebra to A_p.

The root space of the dynamical symmetry group would have dimension m, which is a factor of n. For Lie algebras A_n and D_{2n} (with $n \geq 4$) appear besides E_6 and E_8. For affine Lie algebras $\hat{A}_n$ or $hatD_n$ (with $n \geq 3$) and $\hat{E}_6$, $\hat{E}_7$ and $\hat{E}_8$ appear. For large values of n, there are two alternatives for even values of n.

3. One can also consider quantum arithmetics based on $\oplus$ and $\otimes$ and replace P with its quantum counterpart and solve it in the space of irreps of the finite subgroup G of $U(2)$ defining a quantum analog for an extension of rationals. The roots of the quantum variant of P would be direct sums of irreps of G.

   These quantum roots define nodes of a diagram. This diagram should include as nodes the roots of the Dynkin diagram defined by positive roots, whose number is the dimension n_C of Cartan algebra.

   Could the missing edges correspond to the edges of the Mac-Kay graph in the tensor product with a 2-D representation of $SU(2)$ restricted to a subgroup? The action of 2-D representation would generate the (extended) Dynkin diagram ADE type.

One can look this option in more detail.

1. Assume that adjoint representation *Adj* of an affine or ordinary ADE Lie group L emerges in the tensor product $M^2(C) \otimes ... \otimes M^2(C)$ allowing imbedding of $SU(2)$ as diagonal imbedding. One can imbed the finite subgroup $G \subset SU(2)$ as a diagonal group $G \times G \times ... \times G$ to $M^2(C) \otimes ... \otimes M^2(C)$.

   Also a given representation of G can be embedded as a direct sum of the copies of the representation, each acting in one factor of $M^2(C) \otimes ... \otimes M^2(C)$. The $2 - D$ canonical representation of $G \subset SU(2)$ has a natural action in the $G \times G \times ... \times G$ to $M^2(C) \otimes ... \otimes M^2(C)$ and would generate a McKay graph.

   One can also embed G to L as $G \subset SU(2) \subset L$. *Adj* can be decomposed to irreps G. Therefore the tensor product action of various irreps of G, in particular the canonical 2-D representation, in *Adj* is well-defined. The tensor action of the 2-D canonical representation of G gives a McKay graph such that the nodes have weights telling how many times a given irrep appears in the decomposition of *Adj* to irreps of G. The weighted sum of the dimensions of irreps of G is equal to the dimension of *Adj*.

2. This construction is possible for any Lie group and some consistency conditions should be satisfied. That McKay graph is the same as the generalized Dynkin diagram would be such a consistency condition and leave only simply laced Lie groups.

3. What can one say about the weights of the weighted McKay graph? Could the weights be the number of the images of the positive root under the action of the Weyl group W of L.

   The McKay graph would correspond only to the n_C (dimension of the Cartan algebra) positive roots appearing in the Dynkin diagram of *Adj*. How to continue the Dynkin dynkin to a root diagram of *Adj*?

4. Could the n_C roots in the Dynkin diagram correspond to the roots of a polynomial P in a quantum extension of rationals with roots as irreps of G appearing in the McKay graph. The multiple of a given root would correspond to its orbit under W. The action of W as reflections in the quantum extension of rationals, spanned by the roots of *Adj*, as vectors with integer components would generate all roots of *Adj* as quantum algebraic integers in the quantum extension of rationals.

5. As proposed, one could interpret the Dynkin diagram as a subdiagram of the root diagram of *Adj* and identify its nodes as roots of *Gal* for a suitable polynomial P. The Weyl group could be the minimal transitive subgroup of *Gal*.

6. The Galois group of extension of ... of rationals is a semidirect of Galois groups which can be chosen to be simple so that the polynomials considered are prime polynomials unless one poses additional restrictions. What does this restriction mean for the ADE type Weyl group of assignable to the exrtension

5 Infinite primes as a basic mathematical building block

Infinite primes [K11, K3, K6] are one of the key ideas of TGD. Their precise physical interpretation and the role in the mathematical structure of TGD has however remained unclear.

3 new ideas are be discussed. Infinite primes could define a generalization of the notion of adele; quantum arithmetics could replace $+$ and $\times$ with $\oplus$ and $\otimes$ and ordinary primes with p-adic representations of say HFFs; the polynomial Q defining an infinite prime could be identified with the polynomial P defining the space-time surface: $P = Q$.

5.1 Construction of infinite primes

Consider first the construction of infinite primes [K11].

1. At the lowest level of hierachy, infinite primes (in real sense, p-adically they have unit norm) can be defined by polynomials of the product X of all primes as an analog of Dirac vacuum.

   The decomposition of the simplest infinite primes at the lowest level are of form $aX + b$, where the terms have no common prime divisors. More concretely $a = m_1/n_F$ $b = m_0 n_F$, where n_F is square free integer analogous and the integer m_1 and n_F have no common prime divisors divisors. The divisors of m_2 are divisors of n_F and m_i has interpretation as n-boson state. Power p^k corresponds to k-boson state with momenta p. $n_F = \prod p_i$ has interpretation as many-fermion state satisfying Fermi-Dirac statistics.

   The decomposition of lowest level infinite primes to infinite and finite part has a physical analogy as kicking of fermions from Dirac sea to form the finite part of infinite prime. These states have interpretation as analogs of free states of supersymmetric arithmetic quantum field theory (QFT) There is a temptation to interpret the sum $X/n_F + n_F$ as an analog of quantum superposition. Fermion number is well-defined if one assigns the number of factors of n_F to both n_F and X/n_F.

2. More general infinite primes correspond to polynomials $Q(X) = \sum_n q_n X^n$ required to define infinite integers which are not divisible by finite primes. Each summand $q_n X^n$ must be a infinite integer. This requires that q_n is given by $q_n = m_{B,n}/\prod_{i_1}^n n_{F,i}$ of square free integers $n_{F,i}$ having no common divisors.

   The coefficients $m_{B,n}$ representing bosonic states have no common primes with $\prod n_{F,i}$ and there exists no prime dividing all coefficients $m_{B,n}$: there is no boson with momentum p present in all states in the sum.

   These states have a formal interpretation as bound states of arithmetic supersymmetric QFT. The degree k of Q determines the number of particles in the bound states.

   The products of infinite primes at given level are infinite primes with respect to the primes at the lower levels but infinite integers at their own level. Sums of infinite primes are not in general infinite primes. For instance the sum and difference of $X/n_F + n_F$ and $X/n_F - n_F$ are not infinite primes.

3. At the next step one can form the product of all finite primes and infinite primes constructed in this manner and repeat the process as an analog to second quantization. This procedure can be repeated indefinitely. This repeated quantization a hierarchy of infinite primes, which could correspond to the hierarchy of space-time sheets.

   At the n:th hierarchy level the polynomials are polynomials of n variables X_i. A possible interpretation would be that one has families of infinite primes at the first level labelled by n_1 parameters. If the polynomials $P(x)$ at the first level define space-time surfaces, the interpretation at the level of WCW could be that one has an $n-1$-D surface in WCW parametrized by $n-1$ parameters with rational values and defining a kind of sub-WCW. The WCW spinor fields would be restricted to this surface of WCW.

The Dirac vacuum X brings in mind adele, which is roughly a product of p-adic number fields. The primes of infinite prime could be interpreted as labels for p-adic number fields. Even more generally, they could serve as labels for p-adic representations of various algebras and one could even consider replacing the arithmetic operations with $\oplus$ and $\otimes$ to get the quantum variants of various number fields and of adeles.

The quantum counterparts of nfinite primes at the lowest and also at the higher levels of hierarchy could be seen as a generalization of adeles to quantum adeles.

5.2 Questions about infinite primes

One can ask several questions about infinite primes.

1. Could $\oplus$ and $\otimes$ replace $+$ and $-$ also for infinite primes. This would allow us to interpret the primes p as labels for algebras realized p-adically. This would give rise to quantal counterparts of infinite primes.

2. What could $+ \rightarrow \oplus$ for infinite primes mean physically? Could it make sense in adelic context? Infinite part has finite p-adic norms. The interpretation as direct sum conforms with the fermionic interpretation if the product of all finite primes is interpreted as Dirac sea. In this case, the finite and infinite parts of infinite prime would have the same fermion number.

3. Could adelization relate to the notion of infinite primes? Could one generalize quantum adeles based on $\oplus$ and $\otimes$ so that they would have parts with various degrees of infinity?

5.3 $P=Q$ hypothesis

One cannot avoid the idea that that polynomial, call it $Q(X)$, defining an infinite prime at the first level of the hierarchy, is nothing but the polynomial P defining a 4-surface in M^4 and therefore also a space-time surface. $P=Q$ would be a condition analogous to the variational principle defining preferred extremals (PEs) at the level of H.

There is however an objection.

1. $P=Q$ gives very powerful constraints on Q since it must define an infinite integer. The prime polynomials P are expected to be highly non-unique and an entire class of polynomials of fixed degree characterized by the Galois group as an invariant is in question. The same applies to polynomials Q as is easy to see: the only condition is that powers of a_kX^k defining infinite integers have no common prime factors.

2. It seems that a composite polynomial $P_n \circ ... \circ P_1$ satisfying $P_i = Q_i$ cannot define an infinite prime or even infinite integer. Even infinite integer property requires very special conditions.

3. There is however no need to assume $P_i = Q_i$ conditions. It is enough to require that there exists a composite $P_n \circ ... \circ P_1$ of prime polynomials satisfying $P_n \circ ... \circ P_1 = Q$ defining an infinite prime.

   The physical interpretation would be that the interaction spoils the infinite prime property of the composites and they become analogs of off-mass-shell particles. Exactly this occurs for bound many-particle states of particles represented by P_i represented composite polynomials $P_1 \circ ... P_n$. The roots of the composite polynomials are indeed affected for the composite. Note that also products of Q_i are infinite primes and the interpretation is as a free many-particle state formed by bound states Q_i.

There is also a second objection against $P=Q$ property.

1. The proposed physical interpretation is that the ramified primes associated with $P=Q$ correspond to the p-adic primes characterizing particles. This would mean that the ramimied primes appearing in the infinite primes at the first level of the hierarchy should be physically special.

2. The first naive guess is that for the simplest infinite primes $Q(X) = (m_1/n_F)X + m_2n_F$ at the first level, the finite part m_2n_F has an identification as the discriminant D of the polynomial $P(X)$ defining the space-time surface. This guess has no obvious generalization to higher degree polynomials $Q(X)$ and the following argument shows that it does not make sense.

   Since Q is a rational polynomial of degree 1 there is only a single rational root and discriminant defined by the differences of distinct roots is ill-defined that $Q = P$ condition would not allow the simplest infinite primes.

   Therefore one must give either of these conjectures and since $P = Q$ conjecture dictates the algebraic structure of the quantum theory for a given space-time surface, it is much more attractive.

The following argument gives $P = Q$. One can assign to polynomial P invariants as symmetric functions of the roots. They are invariants under permutation group S_n of roots containing Galois group and therefore also Galois invariants (for polynomials of second order correspond to sum and product of roots appearing as coefficients of the polynomial in the representation $x^2+bx+cx$). The polynomial Q having as coefficients these invariants is the original polynomial. This interpretation gives $P = Q$.

6 Summary of the proposed big picture

In the previous sections the plausible looking building blocks of the bigger picture of the TGD were discussed. Here I try to summarize a guess for the big picture.

6.1 The relation between $M^8 - H$ and $M - M'$ dualities

The first question is whether $M^8 - H$ duality between number theoretical and geometric physics, very probably relating to Langlands duality, corresponds to a duality between M and its commutant M'. Physical intuition suggests that these dualities are independent. M' would more naturally correspond to classical description as dual to quantum description using M. One would assign classical and quantum views to both number theoretic (M^8) and geometric (H) descriptions.

1. At the geometric side M would be realized in terms of HFFs associated with SSA_n, Aff_n and I acting in H. At the number theoretic side, braided Galois group algebras would define the HFFs and have natural action in SSA_n, A_n and I .

2. The descriptions in terms of preferred extremals in H and of polynomials P defining 4-surfaces in M^8 would correspond to classical descriptions. $P = Q$ condition would define preferred polynomials and infinite primes.

3. At the geometric side, M' would correspond to scalar factors of WCW wave functions symplectic invariants identifiable as Kähler magnetic fluxes at both M^4 and CP_2 sectors. They are zero modes and therefore do not contribute to the WCW line element.

4. At the number theoretic side, the wave functions would depend on Galois invariants. Discriminant D, set of roots to which braid strands can be assigned to define n-braid, and ramified primes dividing it in the case of polynomials with rational/integer coefficients are Galois invariants analogous to Kähler fluxes. They code information about the spectrum of virtual mass squared values as roots of P. The strands of braid as Galois invariant correspond to (possibly) monopole flux tubes and one assign them quantized magnetic fluxes as integer valued symplectic invariants.

6.2 Basic mathematical building blocks

The basic mathematical building blocks of quantum aspects of TGD involve at least the following ones.

1. The generalization of arithmetics and even number theory by replacing sum and product by direct sum and tensor product for various algebras and associated representations is a mathematical notion expected to be important and a straightforward generalization of adeles and infinite primes to their quantum counterparts is highly suggestive.

2. Quantum version of adelic physics obtained by replacing ordinary arithmetic operations with direct sum and tensor product relates closely to the fusion of real and various p-adic physics at quantum level.

3. The hierarchy of infinite primes suggested by the many-sheeted space-time suggests a profound generalization of the notion of adelic physics. Infinite primes are defined by polynomials of several variables the basic equation in the general form would be $Q(X_1, ..., X_n) = P(X_1, ..., X_n)$.

6.3 Basic algebraic structures at number theoretic side

Number theoretic side involves several key notions that must have counterparts at the geometric side.

1. Number theoretic side involves Galois groups as counterparts of symplectic symmetries and can be regarded as number theoretic variants of permutation symmetries and lead to the notion of braided Galois group, whose group algebra defines HFF.

2. Galois groups can be decomposed to a hierarchy of normal subgroups, which are simple and therefore primes in group theoretic sense. Simple Galois groups correspond to polynomial primes with respect to functional composition, and one can assign to a given Galois group a set of polynomials with fixed degrees although the polynomials and their order of polynomials in composition are not unique.

3. There is a large class of polynomials giving rise to a given Galois group and they bring in additional degrees of freedom. The variation of the polynomial coefficients corresponding to the same Galois group is analogous to symplectic transformations leaving the induced Kähler form invariant.

   The roots of polynomials define analogs for the strands of n-braid, discriminant D, and ramified primes dividing the discriminant. They are central Galois invariants analogous to Kähler magnetic fluxes at the geometry side.

4. Ramified primes characterize polynomials P but are not fixed by the Galois group, are analogous to the zero modes at the level of H. Magnetic fluxes are their counterparts at the level of H. I have proposed the interpretation of ramified primes p as p-adic primes characterizing elementary particles in the model of particle masses based on p-adic thermodynamics. These primes are rather large: for instance, $M_{127} = 2^{127} - 1$ would characterize electrons. It would however seem that the prime k in SSA_k corresponds to the prime characterizing simple Galois group.

   Also affine algebras Aff_n assignable to the light-like partonic orbits and isometries of H are present and also they appear in p-adic mass calculations based on p-adic thermodynamics. Could the adelic hierarchy p-adic variants of algebras SSA, Aff and I have adelic factors labelled by ramified primes p form also an adelic structure with respect to $\oplus$ and $\otimes$?

6.4 Basic algebraic structures at the geometric side

The symmetry algebras at the level of H define the key quantal structures.

1. The symmetries at the geometric side involve hierarchies A_n of algebras $A_n \in SSA_n, A_n, I_n$ defining hierarchies of factor algebras. The condition that subalgebras A_n and $[A_n, A]$ annihilate physical states gives rise to hierarchies of algebras, which would correspond to those for Galois groups for multiple extensions of rationals. The braided Galois groups for polynomials of degree n n roots/braids would act naturally in A_n so that it would have number theoretic braiding.

2. The decomposition of the Galois group to simple normal subgroups would correspond to a functional composite of prime polynomials, which corresponds to the inclusion hierarchy of HFFs associated with A_n with n identified as the degree of polynomial.

   The polynomials $Q(X)$ defining infinite prime have decomposition to polynomial primes but the polynomial primes in the decomposition cannot define infinite primes.

   Kähler magnetic fluxes for CP_2 and M^4 Kähler forms are symplectic invariants and represent zero modes. At the number theoretic side the discriminant and root spectrum (mass squared spectrum) are classical Galois invariants. States as Galois singlets are Galois invariants at quantum level.

The key equation, not encountered before in the TGD framework, is $P = Q$ motivated by the notion of infinite prime. It would assign to polynomial P unique algebraic structures defining what might be called its quantization. Without this structure one should give up the notion of infinite prime and lose the notion of preferred P as analog of preferred extremal.

REFERENCES

Mathematics

[A1] Langlands program. Available at: `https://en.wikipedia.org/wiki/Langlands_program`.

[A2] Connes A. *Non-commutative Geometry.* Academic Press, San Diego, 1994.

[A3] Beardon AF. Composition factors of polynomials. *Complex Variables, Theory and Application: An International Journal*, 43(3-4):225–239, 2001. Available at: `https://doi.org/10.1080/17476930108815314`.

[A4] Frenkel E. Recent Advances in Langlands program. *AMS*, 41(2):151–184, 2004.

[A5] Frenkel E. Lectures on Langlands program and conformal field theory, 2005. Available at: `https://arxiv.org/abs/hep-th/0512172`.

[A6] Butin F. Branching Law for the Finite Subgroups of $SL(4, C)$. *Ukrainian Mathematical Journal*, 67:1484–1497, 2016. Available at: `https://link.springer.com/article/10.1007/s11253-016-1167-8`.

[A7] Butin F and Perets GS. Branching law for finite subgroups of $SL(3, C)$ and McKay correspondence. *Journal of Group Theory, De Gruyter*, 17(2):191–251, 2013. Available at: `https://hal.archives-ouvertes.fr/hal-00412643/document`.

[A8] Jones FR. *Braid groups, Hecke algebras and type II_1 factors.* 1983.

[A9] Nakamura I. McKay Correspondence, 2002. Available at: `https://www.math.sci.hokudai.ac.jp/~nakamura/McKay071002.pdf`.

[A10] Dixmier J. *Von Neumann Algebras.* North-Holland, Amsterdam, 1981. First published in French in 1957: Les Algebres d'Operateurs dans l'Espace Hilbertien, Paris: Gauthier-Villars.

[A11] Ritt JF. Prime and Composite Polynomials. *Transactions of the American Mathematical Society*, 23(1):51–66, 1922. Available at: `https://doi.org/10.2307/1988911`.

[A12] Khovanov M. MacKay correspondence, Available at: `https://www.math.columbia.edu/~khovanov/finite/QiYou1.pdf`.

[A13] Reid M. McKay correspondence. Available at: `https://arxiv.org/abs/alg-geom/9702016`.

[A14] Lieb EH Temperley NH. Relations between the percolation and colouring problem and other graph-theoretical problems associated with regular planar lattices:some exact results for the percolation problem. *Proc R Soc London*, 322, 1971.

[A15] Jones V. In and around the origin of quantum groups, 2003. Available at: https://arxiv.org/abs/math/0309199.

[A16] Jones VF. Hecke algebra representations of braid groups and link polynomial. *Ann Math*, 126, 1987.

Theoretical Physics

[B1] Minev ZK et al. To catch and reverse a quantum jump mid-flight, 2019. Available at: https://arxiv.org/abs/1803.00545.

Books related to TGD

[K1] Pitkänen M. Construction of WCW Kähler Geometry from Symmetry Principles. In *Quantum Physics as Infinite-Dimensional Geometry*. Available at: https:/tgdtheory.fi/pdfpool/compl1.pdf, 2006.

[K2] Pitkänen M. Identification of the WCW Kähler Function. In *Quantum Physics as Infinite-Dimensional Geometry*. Available at: https:/tgdtheory.fi/pdfpool/kahler.pdf., 2006.

[K3] Pitkänen M. Infinite Primes and Consciousness. In *Mathematical Aspect of Consciousness*. Available at: https:/tgdtheory.fi/pdfpool/infpc.pdf, 2006.

[K4] Pitkänen M. Negentropy Maximization Principle. In *TGD Inspired Theory of Consciousness*. Available at: https:/tgdtheory.fi/pdfpool/nmpc.pdf, 2006.

[K5] Pitkänen M. WCW Spinor Structure. In *Quantum Physics as Infinite-Dimensional Geometry*. Available at: https:/tgdtheory.fi/pdfpool/cspin.pdf, 2006.

[K6] Pitkänen M. Motives and Infinite Primes. In *TGD as a Generalized Number Theory*. Available at: https:/tgdtheory.fi/pdfpool/infmotives.pdf, 2011.

[K7] Pitkänen M. Recent View about Kähler Geometry and Spin Structure of WCW . In *Quantum Physics as Infinite-Dimensional Geometry*. Available at: https:/tgdtheory.fi/pdfpool/wcwnew.pdf, 2014.

[K8] Pitkänen M. About Preferred Extremals of Kähler Action. In *Physics in Many-Sheeted Space-Time: Part I*. Available at: https:/tgdtheory.fi/pdfpool/prext.pdf, 2019.

[K9] Pitkänen M. Evolution of Ideas about Hyper-finite Factors in TGD. In *Hyper-finite Factors and Dark Matter Hierarchy: Part II*. Available at: https:/tgdtheory.fi/pdfpool/vNeumannnew.pdf, 2019.

[K10] Pitkänen M. Langlands Program and TGD. In *TGD as a Generalized Number Theory: Part II*. Available at: https:/tgdtheory.fi/pdfpool/langland.pdf, 2019.

[K11] Pitkänen M. TGD as a Generalized Number Theory: Infinite Primes. In *TGD as a Generalized Number Theory: Part I*. Available at: https:/tgdtheory.fi/pdfpool/visionc.pdf, 2019.

[K12] Pitkänen M. Was von Neumann Right After All? In *Hyper-finite Factors and Dark Matter Hierarchy: Part I*. Available at: https:/tgdtheory.fi/pdfpool/vNeumann.pdf, 2019.

[K13] Pitkänen M. Zero Energy Ontology. In *Towards M-Matrix: part I*. Available at: https:/tgdtheory.fi/pdfpool/ZEO.pdf, 2021.

Articles about TGD

[L1] Pitkänen M. Langlands conjectures in TGD framework. Available at: https:/tgdtheory.fi/public_html/articles/Langlandsagain.pdf., 2011.

[L2] Pitkänen M. Langlands Program and TGD: Years Later. Available at: https:/tgdtheory.fi/public_html/articles/langlandsnew.pdf., 2016.

[L3] Pitkänen M. About number theoretic aspects of NMP. Available at: https:/tgdtheory.fi/public_html/articles/nmpagain.pdf., 2017.

[L4] Pitkänen M. Could McKay correspondence generalize in TGD framework? Available at: https:/tgdtheory.fi/public_html/articles/McKay.pdf., 2017.

[L5] Pitkänen M. Philosophy of Adelic Physics. In *Trends and Mathematical Methods in Interdisciplinary Mathematical Sciences*, pages 241–319. Springer.Available at: https://link.springer.com/chapter/10.1007/978-3-319-55612-3_11, 2017.

[L6] Pitkänen M. Philosophy of Adelic Physics. Available at: https:/tgdtheory.fi/public_html/articles/adelephysics.pdf., 2017.

[L7] Pitkänen M. Copenhagen interpretation dead: long live ZEO based quantum measurement theory! Available at: https:/tgdtheory.fi/public_html/articles/Bohrdead.pdf., 2019.

[L8] Pitkänen M. Some comments related to Zero Energy Ontology (ZEO). Available at: https:/tgdtheory.fi/public_html/articles/zeoquestions.pdf., 2019.

[L9] Pitkänen M. TGD view about McKay Correspondence, ADE Hierarchy, and Inclusions of Hyperfinite Factors. Available at: https:/tgdtheory.fi/public_html/articles/McKay.pdf., 2019.

[L10] Pitkänen M. TGD view about McKay Correspondence, ADE Hierarchy, Inclusions of Hyperfinite Factors, $M^8 - H$ Duality, SUSY, and Twistors. Available at: https:/tgdtheory.fi/public_html/articles/McKaygeneral.pdf., 2019.

[L11] Pitkänen M. A critical re-examination of $M^8 - H$ duality hypothesis: part I. Available at: https:/tgdtheory.fi/public_html/articles/M8H1.pdf., 2020.

[L12] Pitkänen M. A critical re-examination of $M^8 - H$ duality hypothesis: part II. Available at: https:/tgdtheory.fi/public_html/articles/M8H2.pdf., 2020.

[L13] Pitkänen M. The dynamics of SSFRs as quantum measurement cascades in the group algebra of Galois group. Available at: https:/tgdtheory.fi/public_html/articles/SSFRGalois.pdf., 2020.

[L14] Pitkänen M. Zero energy ontology, hierarchy of Planck constants, and Kähler metric replacing unitary S-matrix: three pillars of new quantum theory (short version). Available at: https:/tgdtheory.fi/public_html/articles/kahlersm.pdf., 2020.

[L15] Pitkänen M. Is genetic code part of fundamental physics in TGD framework? Available at: https://tgdtheory.fi/public_html/articles/TIH.pdf., 2021.

[L16] Pitkänen M. Negentropy Maximization Principle and Second Law. Available at: https://tgdtheory.fi/public_html/articles/nmpsecondlaw.pdf., 2021.

[L17] Pitkänen M. Spin Glasses, Complexity, and TGD. https://tgdtheory.fi/public_html/articles/sg.pdf., 2021.

[L18] Pitkänen M. TGD as it is towards the end of 2021. https://tgdtheory.fi/public_html/articles/TGD2021.pdf., 2021.

[L19] Pitkänen M. What could 2-D minimal surfaces teach about TGD? https://tgdtheory.fi/public_html/articles/minimal.pdf., 2021.

[L20] Pitkänen M. About TGD counterparts of twistor amplitudes: part I. https://tgdtheory.fi/public_html/articles/twisttgd1.pdf., 2022.

[L21] Pitkänen M. About TGD counterparts of twistor amplitudes: part II. https://tgdtheory.fi/public_html/articles/twisttgd2.pdf., 2022.

[L22] Pitkänen M. About the number theoretic aspects of zero energy ontology. https://tgdtheory.fi/public_html/articles/ZEOnumber.pdf., 2022.

[L23] Pitkänen M. McKay Correspondence from Quantum Arithmetics Replacing Sum and Product with Direct Sum and Tensor Product? . https://tgdtheory.fi/public_html/articles/McKayGal.pdf., 2022.