The Notion of Generalized Integer

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Abstract

The inspiration for this article came from the article "Space Element Reduction Duplication (SERD) model produces photon-like information packets and light-like cosmological horizons" by Thomas L. Wood expressess the basic assumptions of the SERD approach very coherently and in a systematic way so that it easy easy to criticize them and compare with other views, in my case the TGD view.

The criticism is based on a different interpretation of the discreteness. It would be assignable to cognitive representations based on p-adic numbers fields involving extensions of rationals. Bringing in also the continuous number fields (reals, complex numbers, quaternions, octonions) brings in real space-time as sensory representation and one ends up to a generalization of the standard model proving a number theoretic interpretation for its symmetries.

The approach of Thomas looks to me essentially topological: for instance, the information propagating in the hypergraph is assumed to be topological. In TGD, discrete structures analogs define cognitive representations of the continuous sensory world and are basically number theoretic. The description of the sensory world involves both topology and geometry.

The articulation of this view led to the main result of this article, which is a generalization of the number concept as a fusion of all p-adic number fields and rationals to a single structure that I call generalized integers. Besides being useful in TGD, this framework could be very useful in the modelling of spin glass-like systems.

Contents

1 Introduction

This chapter was inspired by the article "Space Element Reduction Duplication (SERD) model produces photon-like information packets and light-like cosmological horizons" by Thomas L.

Wood, published in Metodologia IV B: Journal of International and Finnish Methodology, expresses the basic assumptions of the SERD approach very coherently and in a systematic way so that it easy to criticize them and compare with other views, in my case the TGD view.

My criticism, summarized below, is based on a different interpretations of the discreteness. In TGD framework would be assignable to cognitive representations based on p-adic numbers fields involving extensions of rationals rather than being a feature of space-time. The introduction of continuous number fields (reals, complex numbers, quaternions, octonions) besides p-adic number fields brings in real space-time as sensory representation and one ends up to a generalization of the standard model proving a number theoretic interpretation for its symmetries.

The approach of Wood looks is essentially topological: for instance, the information propagating in the hypergraph is assumed to be topological and characterize the graph. In TGD, discrete structures analogs define cognitive representations of the continuous sensory world and are basically number theoretic. The description of the sensory world involves both topology and geometry.

1.1 The first reactions to the abstract

The abstract gives a very concise summary of the approach and I have added below my reactions to it. The following commentary is my attempt to understand the basic ideas of SERD. I have also used the third section of the article to clarify my views. I must admit that I didn't quite get the two basic principles in the beginning of the third section. I have slightly re-organized the abstract and hope that I have not done any damage.

[TW] This document describes a correspondence between photons and propagating information packets (PIPs) that are emergent out of the Space Element Reduction Duplication (SERD) model introduced in a rudimentary form in [1, 2]. The SERD model is a discrete background independent microscopic space-time description.

[MP] The assumption of discreteness at the fundamental space-time level raises several challenges. 4-D space-time with Minkowskian signature should somehow emerge. The mere hypergraph might possess under additional assumptions a local dimension defined homologically/combinatorially but would vary. Note that in standard homology theory an embedding to some space is required and would give a metric. Now the distance and other geometric notions look problematic to me. One can also ask what kind of dynamics for hypergraphs could select the 4-D space-time? Should one have a variational principle of some kind?

The notion of symmetries is central in physics. Lorentz invariance or even Poincare invariance should emerge as approximate symmetries at least. Only discrete subgroups of these groups can emerge in the hypergraph approach. Lorentz invariance poses very, perhaps too, powerful constraints on the hypergraphs. The notion of discretized time is introduced. It should be Lorentz invariant and here the light-cone proper time a serves as an analog. a=constant sections would be analogs of hyperbolic 3-space H^3.

[TW] By observation of physically comparable behaviour emerging from this system, through analysis and computer simulation, we draw conclusions of what the form and dynamics of the true underlying space-time may be.

By treating elements of the system as fundamental observers, mathematical and empirical evidence is obtained of the existence of fully emergent light-like cosmological horizons, implying the existence of causally separated 'pocket universes'.

[MP] The emergence of the analogy with expanding cosmology presumably reflects the underlying dynamics implying the increase of the size of the hypergraph. The emergence of light-like causal horizons is natural if the dynamics involves maximal velocity of propagation for the signals. This is probably due to the locality of the basic dynamics involving only local changes of the hypergraph topology. Locality and classicality raise challenges if one wants to describe phenomena like quantum entanglement.

[TW] The SERD model is a hypergraph of connected hyperedges called Point Particles (PP) which represent the fundamental constituents of all matter and particles (and therefore observers) separated by strings of consecutive and fundamental elements or edges called Space Elements (SE).

[MP] I had to clarify myself what a hypergraph is. Hypergraph is a generalization of graphs. Also it contains the set of vertices/nodes. The notion of edge connecting a pair of vertices is however generalized to a hyperedge (PP) as a pair of subsets of vertices. PPs correspond to hyperedges as fundamental constituents of matter and formed by pairs of subsets of the set of nodes.

One could interpret this as a combinatorial counterpart for a length scale hierarchy of TGD in which a set can be approximated as a point. One might also interpret subsets of vertices as analogs of bound states of fundamental particles. In the TGD framework, many-sheeted space-time and various other hierarchies serve as its analogs.

Space elements (SEs) would bring in basic aspects of 3-space. It is said that they are infinitesimal or maximally small. SEs would be like edges (not hyperedges) of the hypergraph. Consecutive SEs in turn form interaction edges (IEs) connecting PPs. IEs store and transmit information relating to the structure space. What comes to mind is that functionally PPs are like neutrons and neuron groups and IEs are like axons.

[TW] All elements are separated by nodes called Information Gaps (IGs), that store propagating topological information of the hypergraph. Information gaps (IGs) are between PPs and SEs, between SEs and between PPs themselves.

[MP] What distinguishes the SERD model from physical theories, is that information takes the role of matter. Information is treated as some kind of substance. The basic objection is that conscious information is always about something, whereas matter just is.

IGs have the role of interfaces somewhat analogous to black-hole horizon assumed to store information in the holographic picture. One could see PPs as the nodes and IEs as the edges or SEs as the edges and IGs as the nodes. IGs could have synaptic contacts as analogs.

[TW] In time step (TS), SE can duplicate and reduce (disappear) while the PPs split and merge through discrete time. These processes create space or destroy it and increase or reduce the effective distance between PPs. Splitting generates an SE between the resulting PPs. These are known as the actions of the elements and create a highly dynamic multi-way system.

[MP] Time step (TS) is a further basic notion and corresponds to an elementary event as nearest neighbor interaction taking during the time chronon. The propagation rate for information is CS/TS and is analogous to maximal signal velocity. The counterpart of the space-time metric is thus brought in by the introduction of TS and CS.

SEs emerge or disappear so that the effective distances of the nearby points change: this would the counterpart for the dynamics of space-time metric in General Relativity. I understood that duplication and reduction effectively corresponds to the duplication or halving of the distance assignable to SE.

[TW] Elements have an 'awareness' of the information around them and communicate with their nearest neighbours through time.

[MP] The treatment of elements as fundamental observers is an interesting idea but can be criticized. Why not PPs? One could also argue that the SEs become conscious observers only under some additional assumptions. For instance, one can imagine that they represent matter and become fundamental conscious observers if fermions or fermion pairs can be assigned to them.

The abstract says nothing about quantum theory. To my view it is very difficult to imagine how quantum theory could emerge from an approach based on classical probability and some kind of quantum approach would be required to understand entanglement and state function reduction.

1.2 Plan of the article

In the sequel TGD view of the discretization interpreted as cognitive representation is described. The surprise was the discovery of what I call generalized integers and rationals as a union of various p-adic number fields with different p-adic number fields glued together along numbers which belong to both p-adic number fields. I do not know whether mathematicians have played with this thought. This space has an ultrametric topology and could have application to the description of spin glass type systems [L6]. In TGD it could have application in the mathematical description of processes in which the p-adic prime associated with the particle changes.

2 Fundamental discretization as a cognitive representation?

Something is discrete at the fundamental level: is it space-time or only a discrete cognitive representation, a discretization of a continuous space-time? The essential assumption of

SERD is that it is space-time, which is fundamentally discrete and realized as hypergraph. The basic problem is that it is not clear whether the notions of space-time dimensions, distance, angle, and curvature can emerge in a purely combinatorial approach in which only distance between nearby nodes is a metric notion. These notions also have a formal generalization to gauge theories.

The alternative approach would be based on the observation that cognition is discrete and finite. Cognition provides representations of the physical world. Could one assume that the physical world has continuous geometry and that only cognition is discrete?

2.1 Could the cognitive Universe consist of generalized integers?

Integers (and rationals) are the simplest discrete but infinite systems. Integers/rationals are usually assumed to have real topology. One can however imagine an infinite number of p-adic topologies, which are ultrametric and are defined by a p-adic norm having values coming as powers of prime p. p-Adic primes typically have an infinite expansion in powers of p and large powers of p have small p-adic norm in contrast to the real norm.

p-Adic integer/rational has expansion in powers of p and the inverse of the smallest power in the expansion determines the norm so that the notion of size is completely different for p-adic and real integers. Note that also the p-adic expansion of rationals involves an infinite number of powers of p but is periodic. p-Adic transcendentals do not have this property. Note also that p-adic integers modulo p define a finite field $G(p)$.

p-Adic integers are only weakly ordered. Only if two p-adic integers/ rationals have different p-adic norms, can one tell which is the larger one. One can however construct continuous maps from p-adics to reals to approximately preserve the norm. p-Adic norm is ultrametric and this property is essential in the thermodynamic models of spin glass energy landscape [L6].

One could, at least as the first guess, imagine that the Universe of cognition consists of integers/rationals or a finite subset of them and that one also allows integers/rationals, which are infinite as real integers but finite as p-adic integers for some prime p.

One can decompose generalized integers to subsets with different p-adic topologies.

1. Regions corresponding to two different p-adic topologies p_1 and p_2 have as an interface as the set integers, which have an expansion in powers of $n_{12} = p_1p_2$. Therefore the cognitive world decomposes into p-adic regions having interfaces, which consist of power series of $n_{12..k} = p_1^{k_1} ... \times p_k^{n_k}$. Ordinary integer n with a decomposition to primes belongs to the interface of the p-adic worlds corresponding to the prime factors.

   How does this decomposition relate to adeles [L2, L1], which can be regarded as a Cartesian product of p-adic number fields defining and of reals? Adeles correspond to a Cartesian product but now one has a union so that these concepts seems to be different. I do not know whether mathematicians have encountered the notion of generalized integers and rationals.

2. Each p-adic region decomposes into shells, kinds of analogs of mass shells, consisting of p-adic integers with p-adic norm given by a power of p.

3. The distance between the points of the cognitive sub-landscape corresponding to p would be defined by the p-adic norm. The points with the same p-adic norm would have a distance defined as the p-adic norm of their difference. This distance is the same for several point pairs so that p-adic topology is much rougher than the real topology. For instance the p-adic norm of numbers $1, ..., p-1$ is the same.

4. One could define a distance between points associated with p-adic topologies p_1 and p_2 as the shortest distance between them identified as the sum of the distances to the interface between these regions.

In this framework, the analog of a hypergraph would be simply a subset of generalized integers decomposing to p-adic integers labeled by some subset of primes.

1. The simplest dynamical operation, having now an interpretation as a cognitive operation, would be addition or removal of a p-adic integer corresponding to some value of p-adic prime or several of them. The addition would have an interpretation or worsening or improving the cognitive representation for some prime p.

2. Arithmetic operations for the points inside a region corresponding to a given p are possible. Arithmetic operations of finite integers are basic elements of at least human cognition and their sum and product would correspond to "particle reactions" in which two points fuse together to form a sum or product. If infinite integers can be expressed as power series of integers n_1 and n_2, they can be regarded as p-adic integers for the factors of n_1 and n_2 and both sum and product make sense for common prime factors. Note that the operations are well-defined also for generalized rationals.

3. What happens in the arithmetic operations information theoretically? In the product operation, the outcome is in the interface region associated with n_1 and n_2 and the information about factors is not lost since a measurement revealing prime factors can be done repeatedly.

   The projection operator applied to a quantum superposition of integers would project to a subspace of integers, which are divisible by a given prime p. This operation could be repeated for different primes and eventually give the prime number decomposition for some integer n in the superposition.

   One strange fact about idiot savants described by Oliver Sacks (this is discussed from the TGD point of view in [K1]) is that they can decompose integers into prime factors and obviously see the emergence of the prime factors. Could this kind of cognitive measurement be in question?

   Sum does not in general belong to the interface region of either integer and information is lost since many number pairs give rise to the same sum. Therefore sum and product are information-theoretically very different operations.

Could there be a quantum physical realization for the arithmetic operations? Could they relate to our conscious arithmetic thinking?

1. Consider first the sum operation. Quantum numbers, such as momenta, represented as integers or even algebraic integers are conserved in the physical reaction vertices. The conserved quantum numbers for the final state for a fusion reaction are sums of integers so that these reactions have an arithmetic interpretation.

2. In the case of a product, the fusion reaction should give a product of integers n_1 and n_2 or a representation of it? One should have conserved multiplicative quantum numbers in the vertex.

   Phase factors as eigenvalues of unitary operators are such. They should form a multiplicative group as representation of integers or even rationals. Integer scalings define such a group. One can also consider eigenvalues $n^{i\phi}$, ϕ some fixed phase angle. The operator would therefore be a scaling represented unitarily by these phase factors.

   Initial state would be a product of eigenstates of the scaling operator with eigenphases $n_1^{i\phi}$ and $n_2^{i\phi}$ and the final state would be a single particle state with the eigenvalue $n_1^{i\phi}n_2^{i\phi} = (n_1n_2)^{i\phi}$. One can say that n_1 acts on n_2 by scaling or vice versa. Interestingly, at the fundamental level scalings replace time translations in the TGD framework (and also in superstring theory), and this is especially so for spin glass phase [L6].

   Interestingly, sum appears at the level of Lie algebras and product at the level of Lie groups.

In quantum groups also the reverse operations, co-product and co-sum, having pair creation as analog, are possible. For the co-sum the information increases for the product. These operations would be time reversals of each other. In the zero energy ontology (ZEO) of TGD time reversal occurs in "big" (ordinary) state function reductions (BSFRs) [L3, L8] [K2]. What comes to mind is that the idiot savants described by Sacks might perform a time reversal decomposing product to prime factors. The cognitive measurement would correspond to BSFR.

Note that ZEO also predicts "small" state function reductions (SSFRs), which do not change the arrow of time and give rise to the flow of consciousness whereas BSFR corresponds to a universal counterpart of death or of falling asleep. It is the TGD counterpart of repeated measurements in the Zeno effect and of weak measurements of quantum optics.

This cognitive world would in TGD correspond physically to the most general spin glass energy landscape having an ultrametric topology [L6].

2.2 The algebraic extensions of p-adic number fields are discrete

The proposed structure does not have any natural notion of dimension. We are however able to cognize higher dimensional spaces using formulas.

1. p-Adic number fields indeed allow infinite hierarchies of algebraic extensions obtained by adding to them roots of polynomials, which are algebraic numbers. These induce extensions of p-adic number fields as finite fields $G(p,k)$ having algebraic dimension, which is at most the dimension of the corresponding extensions of rationals.

2. It is natural to assume that cognitive representations are always finite. This suggests that the set of "populated" points of the cognitive space is discrete and even finite. Being "populated" could mean that a fermion, having an interpretation as a generator of Boolean algebra, is labelled by the algebraic number defining the point. In a more general formulation bringing in quaternions and octonions as number fields: algebraic complexified quaternions would define the momentum components of fermions.

   What has been said above, generalizes almost as such and one obtains a hierarchy of generalized integers as algebraic extensions of generalized integers at the lowest level. This could generalize the rational number based computationalism (Turing paradigm) to an entire hierarchy of cognitive computationalisms. The hierarchy of algebraic extensions suggests the same.

3. The algebraic complexity of generalized integers increases with the dimension of extension and in the TGD framework it corresponds to an evolutionary hierarchy. The dimension of extension defines what is identified in terms of an effective Planck constant.

2.3 But what about the real world?

A hierarchy of p-adicities and hierarchies of the algebraic extensions of p-adicities have been obtained. The 4-D world of sensory perceptions with its fundamental symmetries is however still missing. Could number theory come to rescue also here? This is indeed the case.

1. The fundamental continuous number fields consist of reals, complex numbers, quaternions and octonions with dimensions 1,2,4, 8 [L4, L5, L9]. Quaternions cannot as such correspond to 4-D space-time since the number theoretic purely algebraic norm defines the Euclidean metric.

2. This norm can be however algebraically continued to the complexification of quaternions obtained by adding a commuting imaginary unit i commuting with quaternionic and octonionic imaginary units. This algebraic norm squared does not involve complex conjugation as the Hilbert space norm and is in general complex but real for the subspaces corresponding to various metric signatures (a given component of quaternion are either real or imaginary). One obtains therefore Minkowski space and even more: its variants with various metric signatures.

3. One can imagine a generalization of the notion of generalized integer so that one would have hierarchies of generalized complex numbers, quaternions and octonions and their complexifications for various extensions of rationals.

   A possible problem relates to the p-adic variants of quaternions, octonions and complex numbers. Consider the inverse $z^{-1} = (x-iy)/(x^2+y^2)$ of p-adic complex numbers $z = x+iy$. The problem is that $x^2 + y^2$ can vanish since there is no notion of sign of the number. For $p\ mod\ 4 = 1$, $\sqrt{-1}$ is an ordinary p-adic number, albeit with an infinite pinary expansion so that for $y = \sqrt{-1}x$, one has this problem.

   Could the finiteness of cognition solve the problem? If only finite p-adic integers and rationals can define momentum components of fermions (finite cognitive and measurement resolution), the problem disappears.

   Could one give up the field property for the p-adic variants of classical number fields? Already the complexification by i forces to give up the field property but has physical meaning since it makes Minkowski signature possible.

This would give Minkowski space M^4 as a special case. This is however not enough. One wants curved 4-D space-times. The basic structure is complexified octonions.

1. One should obtain 4-D surfaces of M^8 generalizing empty Minkowski space M^4. Octonions fail to be associative and at the level of M_c^8 the natural proposal is that there is number theoretic dynamics based on associativity. The 4-D surfaces must be associative in some sense. The geometric vision predicts holography and this holography should have a number theoretic counterpart based on associativity.

2. The first guess is that the tangent space of 4-surface is associative and thus quaternionic. This gives only M^4 and is therefore trivial [L4, L5, L9].

   The requirement that the normal space of the 4-surface Y^4 in M_c^8 is associative/quaternionic however works. If one requires that the normal subspace contains also a commutative (complex) subspace, one ends up to M^8-H-duality ($H=M^4\times CP_2$ mapping the associative 4-D surfaces Y^4 of M_c^8 to space-time surfaces X^4 in H determined by holography forced by generalized coordinate invariance. The symmetries of H include Poincare symmetries and standard model symmetries.

3. At the level of M^8, associativity of the normal space allows also 6-D surfaces with 2-D commutative normal space and they can be interpreted in terms of analogs of 6-D twistor spaces of 4-D surfaces Y^4. They can be mapped to to the twistor space of H by M^8-H duality and define 6-D twistor spaces of space-time surfaces X^4 of H. What is beautiful is that the Kähler structure for the twistor space of H exists only for the choice $H=M^4\times CP_2$, which is also forced by the associative dynamics [?]! TGD is unique!

4. The dynamics would rely on holography but how to get the algebraic extensions? The roots of a polynomial P with rational or even integer coefficients satisfying some additional conditions would define the needed extension of rationals. The roots would in the general case define complex mass shells H_c^3 as complex variants of hyperbolic 3-spaces H^3 in $M_c^4\subset M_c^8$ having interpretation as a momentum space. M^8-H duality serves as a generalization of momentum position duality. The 3-D surfaces as subsets of these H^3:s define the data of the associative holography and are contained by the 4-surface Y^4.

5. Cognitive representation would be defined as a unique number theoretic discretization of the 4-surface Y^4 of M_c^8 consisting of points, whose number theoretically preferred linear Minkowski coordinates are algebraic integers in an extension defining the 4-surface in question. This discretization induces discretization of the space-time surface via M^8-H duality. The cognitive representations are number-theoretically universal and belong to the intersections of realities and p-adicities.

6. The mass shells H_c^3 are very special since in the preferred Minkowski coordinates a cognitive explosion takes place. All algebraic rationals, in particular integers, are points of H_c^3. Algebraic integers are physically favored and define components of four-momenta. Galois confinement [L7] states that the total momenta have components which are ordinary integers when a suitable momentum unit is used.

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