What p-adic icosahedron could mean? And what about p-adic manifold?

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Abstract

The original focus of this article was p-adic icosahedron. The discussion of attempt to define this notion however leads to the challenge of defining the concept of p-adic sphere, and more generally, that of p-adic manifold, and this problem soon became the main target of attention since it is one of the key challenges of also TGD.

There exists two basic philosophies concerning the construction of both real and p-adic manifolds: algebraic and topological approach. Also in TGD these approaches have been competing: algebraic approach relates real and p-adic space-time points by identifying the common rationals. Finite pinary cutoff is however required to achieve continuity and has interpretation in terms of finite measurement resolution. Canonical identification maps p-adics to reals and vice versa in a continuous manner but is not consistent with p-adic analyticity nor field equations unless one poses a pinary cutoff. It seems that pinary cutoff reflecting the notion of finite measurement resolution is necessary in both approaches. This represents a new notion from the point of view of mathematics.

1. One can try to generalize the theory of real manifolds to p-adic context. The basic problem is that p-adic balls are either disjoint or nested so that the usual construction by gluing partially overlapping spheres fails. This leads to the notion of Berkovich disk obtained as a completion of p-adic disk having path connected topology (non-ultrametric) and containing p-adic disk as a dense subset. This plus the complexity of the construction is heavy price to be paid for path-connectedness. A related notion is Bruhat-Tits tree defining kind of skeleton making p-adic manifold path connected. The notion makes sense for the p-adic counterparts of projective spaces, which suggests that p-adic projective spaces (S^2 and CP_2 in TGD framework) are physically very special.
2. Second approach is algebraic and restricts the consideration to algebraic varieties for which also topological invariants have algebraic counterparts. This approach looks very natural in TGD framework - at least for imbedding space. Preferred extremals of Kähler action can be characterized purely algebraically - even in a manner independent of the action principle - so that they might make sense also p-adically.

Number theoretical universality is central element of TGD. Physical considerations force to generalize the number concept by gluing reals and various p-adic number fields along rationals and possible common algebraic numbers. This idea makes sense also at the level of space-time and of "world of classical worlds" (WCW).

Algebraic continuation between different number fields is the key notion. Algebraic continuation between real and p-adic sectors takes place along their intersection which at the level of WCW correspond to surfaces allowing interpretation both as real and p-adic surfaces for some value(s) of prime p. The algebraic continuation from the intersection of real and p-adic WCWs is not possible for all p-adic number fields. For instance, real integrals as functions of parameters need not make sense for all p-adic number fields. This apparent mathematical weakness can be however turned to physical strength: real space-time surfaces assignable to elementary particles can correspond only some particular p-adic primes. This would explain why elementary particles are characterized by preferred p-adic primes. The p-adic prime determining the mass scale of the elementary particle could be fixed number theoretically rather than by some dynamical principle formulated in real context (number theoretic anatomy of rational number does not depend smoothly on its real magnitude!).

Although Berkovich construction of p-adic disk does not look promising in TGD framework, it suggests that the difficulty posed by the total disconnectedness of p-adic topology is real. TGD in turn suggests that the difficulty could be overcome without the completion to a non-ultrametric topology. Two approaches emerge, which ought to be equivalent.

1. The TGD inspired solution to the construction of path connected effective p-adic topology is based on the notion of canonical identification mapping reals to p-adics and vice versa in a continuous manner. The trivial but striking observation was that canonical identification satisfies triangle inequality and thus defines an Archimedean norm allowing to induce real topology to p-adic context. Canonical identification with finite measurement resolution defines chart maps from p-adics to reals and vice versa and preferred extremal property allows to complete the discrete image to hopefully space-time surface unique within finite measurement resolution so that topological and algebraic approach are combined. Finite resolution would become part of the manifold theory. p-Adic manifold theory would also have interpretation in terms of cognitive representations as maps between realities and p-adicities.
2. One can ask whether the physical content of path connectedness could be also formulated as a quantum physical rather than primarily topological notion, and could boil down to the non-triviality of correlation functions for second quantized induced spinor fields essential for the formulation of WCW spinor structure. Fermion fields and their n-point functions could become part of a number theoretically universal definition of manifold in accordance with the TGD inspired vision that WCW geometry - and perhaps even space-time geometry - allow a formulation in terms of fermions. This option is a mere conjecture whereas the first one is on rigorous basis.

Contents

1 Introduction

This article was originally meant to be a summary of what I understand about the article "The p-Adic Icosahedron" in Notices of AMS [A8]. The original purpose was to summarize the basic ideas and discuss my own view about more technical aspects - in particular the generalization of Riemann sphere to p-adic context which is rather technical and leads to the notion of Bruhat Tits tree and Berkovich space. About Bruhat-Tits tree there is a nice web article titled p-Adic numbers and Bruhat-Tits tree [A3] describing also basics of p-adic numbers in a very concise form.

The notion of p-adic icosahedron leads to the challenge of constructing p-adic sphere, and more generally p-adic manifolds and this extended the intended scope of the article and led to consider the fundamental questions related to the construction of TGD.

Quite generally, there are two approaches to the construction of manifolds based on algebra *resp.* topology.

1. In algebraic geometry manifolds - or rather, algebraic varieties - correspond to solutions of algebraic equations. Algebraic approach allows even a generalization of notions of real topology such as the notion of genus.

2. Second approach relies on topology and works nicely in the real context. The basic building brick is n-ball. More complex manifolds are obtained by gluing n-balls together. Here inequalities enter the game. Since p-adic numbers are not well-ordered they do not make sense in purely p-adic context unless expressed using p-adic norm and thus for real numbers. The notion of boundary is also one of the problematic notions since in purely p-adic context there are no boundaries.

1.1 The attempt to construct p-adic manifolds by mimicking topological construction of real manifolds meets difficulties

The basic problem in the application of topological method to manifold construction is that p-adic disks are either disjoint or nested so that the standard construction of real manifolds using partially overlapping n-balls does not generalize to the p-adic context. The notions of Bruhat-Tits tree, building, and Berkovich disks and Berkovich space are represent attempts to overcome this problem. Berkovich disk is a generalization of the p-adic disk obtained by adding additional points so that the p-adic disk is a dense subset of it. Berkovich disk allows path connected topology which is not ultrametric. The generalization of this construction is used to construct p-adic manifolds using the modification of the topological construction in the real case. This construction provides also insights about p-adic integration.

The construction is highly technical and complex and pragmatic physicist could argue that it contains several un-natural features due to the forcing of the real picture to p-adic context. In particular, one must give up the p-adic topology whose ultra-metricity has a nice interpretation in the applications to both p-adic mass calculations and to consciousness theory.

I do not know whether the construction of Bruhat-Tits tree, which works for projective spaces but not for Q_p^n (!) is a special feature of projective spaces, whether Bruhat-Tits tree is enough so that no completion would be needed, and whether Bruhat-Tits tree can be deduced from Berkovich approach. What is remarkable that for $M^4 \times CP_2$ p-adic S^2 and CP_2 are projective spaces and allow Bruhat-Tits tree. This not true for the spheres associated with the light-cone boundary of $D \neq 4$-dimensional Minkowski spaces.

1.2 Two basic philosophies concerning the construction of p-adic manifolds

There exists two basic philosophies concerning the construction of p-adic manifolds: algebraic and topological approach. Also in TGD these approaches have been competing: algebraic approach relates real and p-adic space-time points by identifying common rationals. Finite pinary cutoff is however required to achieve continuity and has interpretation in terms of finite measurement resolution. Canonical identification maps p-adics to reals and vice versa in a continuous manner but is not consistent with field equations without pinary cutoff.

1. One can try to generalize the theory of real manifolds to p-adic context. Since p-adic balls are either disjoint or nested, the usual constuction by gluing partially overlapping balls fails. This leads to the notion of Berkovich disk obtained as a completion of p-adic disk having path connected topology (non-ultrametric) and containing p-adic disk as a dense subset. This plus the complexity of the construction is heavy price to be paid for path-connectedness. A related notion is Bruhat-Tits tree defining kind of skeleton making p-adic manifold defining its boundary path connected. The notion makes sense for the p-adic counterparts of projective spaces, which

suggests that p-adic projective spaces (S^2 and CP_2 in TGD framework) are physically very special.

2. Second approach is algebraic and restricts the consideration to algebraic varieties for which also topological invariants have algebraic counterparts. This approach is very natural in TGD framework, where preferred extremals of Kähler action can be characterized purely algebraically - even in a manner independent of the action principle - so that they make sense also p-adically.

   At the level of WCW algebraic approach combined with symmetries works: the mere existence of Kähler geometry implies infinite-D group of isometries and fixes the geometry uniquely. One can say that infinite-D geometries are the final victory of Erlangen program. At space-time level it however seems that one must have correspondence between real and p-adic worlds since real topology is the "lab topology". Canonical identification should enter the construction.

1.3 Number theoretical universality and the construction of p-adic manifolds

Construction of p-adic counterparts of manifolds is also one of the basic challenges of TGD. Here the basic vision is that one must take a wider perspective. One must unify real and various p-adic physics to single coherent whole and to relate them. At the level of mathematics this requires fusion of real and p-adic number fields along common rationals and the notion of algebraic continuation between number fields becomes a basic tool.

The number theoretic approach is essentially algebraic and based on the gluing of reals and various p-adic number fields to a larger structure along rationals and also along common algebraic numbers. A strong motivation for the algebraic approach comes from the fact that preferred extremals [K1, K13] are characterized by a generalization of the complex structure to 4-D case both in Euclidian and Minkowskian signature. This generalization is independent of the action principle. This allows a straightforward identification of the p-adic counterparts of preferred extremals. The algebraic extensions of p-adic numbers play a key role and make it possible to realize the symmetries in the same manner as they are realized in the construction of p-adic icosahedron.

The lack of well-ordering of p-adic numbers implies strong constraints on the formulation of number theoretical universality.

1. The notion of set theoretic boundary does not make sense in purely p-adic context. Quite, generally everything involving inequalities can lead to problems in p-adic context unless one is able to define effective Archimedean topology in some natural manner. Canonical identifcation inducing real topology to p-adic context would allow to achieve this.

2. The question arises about whether real topological invariants such as genus of partonic 2-surface make sense in the p-adic sector: for algebraic varieties this is the case. One would however like to have a more general definition and again Archimedean effective topology is suggestive.

3. Integration poses problems in p-adic context and algebraic continuation from reals to p-adic number fields seems to be the only possible option making sense. The continuation is however not possible for all p-adic number fields for given surface. This has however a beautiful interpretation explaining why real space-time sheets (and elementary particles) are characterized by some p-adic prime or primes. The p-adic prime determining the mass scale of the elementary particle could be fixed number theoretically rather than by some dynamical principle formulated in real context (number theoretic anatomy of rational number does not depend smoothly on its real magnitude!). A more direct approach to integration could rely on canonical integration as a chart map allowing to define integral on the real side.

4. Only those discrete subgroups of real symmetries, which correspond matrices with elements in algebraic extension of p-adic numbers can be realized so that a symmetry breaking to discrete subgroup consistent with the notion of finite measurement resolution and quantum measurement theory takes place. p-Adic symmetry groups can be identified as unions of elements of discrete subgroup of the symmetry group (making sense also in real context) multiplied by a p-adic variant of the continuous Lie group. These genuinely p-adic Lie groups are labelled by powers of p telling the maximum norm of the Lie-algebra parameter. Remarkably, effective values of

Planck constant come as powers of p. Whether this interpretation for the hierarchy of effective Planck constants is consistent with the interpretation in terms of n-furcations of space-time sheet remains an open question.

1.4 How to achieve path connectedness?

The basic problem in the construction of p-adic manifolds is the total disconnectedness of the p-adic topology implied by ultrametricity. This leads also to problems with the notion of p-adic integration. Physically it seems clear that the notion of path connectedness should have some physical counterpart.

The notion of open set makes possib le path connectedness possible in the real context. In p-adic context Bruhat-Tits tree and Berkovich disk are introduced to achieve the same goal. One can of course ask whether Berkovich space could allow to achieve a more rigorous formulation for the p-adic counterparts of CP_2, of partonic 2-surfaces, their light-like orbits, preferred extremals of Kähler action, and even the "world of classical worlds" (WCW) [K8, K2]. To me this construction does not look promising in TGD framework but I could be wrong.

TGD suggests two alternative approaches to the problem of path connectedness. They should be equivalent.

1.4.1 p-Adic manifold concept based on canonical identification

The TGD inspired solution to the construction of path connectd p-adic topology is based on the notion of canonical identification mapping reals to p-adics and vice versa in a continuous manner.

1. Canonical identification is used to map the predictions of p-adic mass calculations to map the p-adic value of the mass squared to its real counterpart. It makes also sense to map p-adic probabilities to their real counterparts by canonical identification. In TGD inspired theory of consciousness canonical identification is a good candidate for defining cognitive representations as representations mapping real preferred extremals to p-adic preferred extremals as also for the realization of intentional action as a quantum jump replacing p-adic preferred extremal representing intention with a real preferred extremal representing action. Could these cognitive representations and their inverses actually define real coordinate charts for the p-adic "mind stuff" and vice versa?

2. The trivial but striking observation was that it satisfies triangle inequality and thus defines an Archimedean norm allowing to induce real topology to p-adic context. Canonical identification with finite measurement resolution defines chart maps from p-adics to reals (rather than p-adics!) and vice versa and preferred extremal property allows to complete the discrete image to hopefully unique space-time surface so that topological and algebraic approach are combined. Without preferred extremal property one can complete to smooth real manifold (say) but the completion is much less unique - which indeed conforms with finite pinary resolution.

3. Also the notion of integration can be defined. If the integral for - say- real curve at the map leaf exists, its value on the p-adic side for its pre-image can be defined by algebraic continuation in the case that it exists. Therefore one can speak about lengths, volumes, action integrals, and similar things in p-adic context. One can also generalize the notion of differential form and its holomomorphic variant and their integrals to the p-adic context. These generalizations allow a generalization of integral calculus required by TGD and also provide a justification for some basic assumptions of p-adic mass calculations.

1.4.2 Could path connectedness have quantal description?

The physical content of path connectedness might also allow a formulation as a quantum physical rather than primarily topological notion, and could boil down to the non-triviality of correlation functions for second quantized induced spinor fields essential for the formulation of WCW spinor structure. Fermion fields and their n-point functions could become part of a number theoretically universal definition of manifold in accordance with the TGD inspired vision that WCW geometry - and perhaps even space-time geometry - allow a formulation in terms of fermions.

The natural question of physicist is whether quantum theory could provide a fresh number theoretically universal approach to the problem. The basic underlying vision in TGD framework is that second quantized fermion fields might allow to formulate the geometry of "world of classical worlds" (WCW) (for instance, Kähler action for preferred extremals and thus Kähler geometry of WCW would reduce to Dirac determinant [K7]). Maybe even the geometry of space-time surfaces could be expressed in terms of fermionic correlation functions.

This inspires the idea that second quantized fermionic fields replace the K-valued (K is algebraic extension of p-adic numbers) functions defined on p-adic disk in the construction of Berkovich. The ultrametric norm for the functions defined in p-adic disk would be replaced by the fermionic correlation functions and different Berkovich norms correspond to different measurement resolutions so that one obtains also a connection with hyper-finite factors of type II_1. The existence of non-trivial fermionic correlation functions would be the counterpart for the path connectedness at space-time level. The 3-surfaces defining boundaries of a connected preferred extremal are also in a natural manner "path connected": the "path" is defined by the 4-surface. At the level of WCW and in zero energy ontology (ZEO) [K12] WCW spinor fields are analogous to correlation functions having collections of these disjoint 3-surfaces as arguments. There would be no need to complete p-adic topology to a path connected topology in this approach.

It must be emphasized that this apporach should be consistent with the first option and that it is much more speculative that the first option.

1.5 About literature

It is not easy to find readable literature from these topics. The Wikipedia article about Berkovich space is written with a jargon giving no idea about what is involved. There are video lectures [A6] about Berkovich spaces. The web article about Berkovich spaces by Temkin [A13] seems too technical for a non-specialist. The slides [A14] however give a concise bird's eye of view about the basic idea behind Berkovich spaces.

1.6 Topics of the article

The article was originally meant to discuss p-adic icosahedron. Although the focus was redirected to the notion of p-adic manifold - especially in TGD framework - I decided to keep the original starting point since it provides a concrete manner to end up with the deep problems of p-adic manifold theory and illustrates the group theoretical ideas.

- In the first section icosahedron is described in real context. In the second section the ideas related to its generalization to the p-adic context are introduced. After that I discuss how to define sphere in p-adic context.
- In the section about algebraic universality I consider the problems related to the challenge of defining p-adic manifolds TGD point of view, which is algebraic and involves the fusion of various number fields and number theoretical universality as additional elements.
- The key section of the article describes the construction of p-adic space-time topology relying on chart maps of p-adic preferred extremals defined by canonical identification in finite measurement resolution and on the completion of discrete chart maps to real preferred extremals of Kähler action. The needed path-connected topology is the topology induced by canonical identification defining real chart maps for p-adic space-time surface. Canonical identification allows also the definition of p-adic valued integrals and definition of p-adic differential forms crucial in quqantum TGD.
- Last section discusses in rather speculative spirit the possibility of defining space-time surfaces in terms of correlation functions of induced fermion fields.

2 Real icosahedron and its generalization to p-adic context

I summarize first the description of icosahedron in real context allowing a generalization to the p-adic context and consider the the problems related to the precise definition of p-adic icosahedron.

2.1 What does one mean with icosahedron in real context?

The notion of icosahedron [A1] is a geometric concept involving the notion of distance. In p-adic context this notion does not make sense since one cannot calculated distances, between points using standard formulas. Same applies to areas and volumes. The reason is that Riemann integral does not generalize and this is due to the fact that p-adic numbers are not well-ordered: one cannot say whether for two p-adic numbers of same norm $a < b$ or $b < a$ holds true.

Platonic solids [A4] are however characterized by their isometry groups and group theory makes sense also in p-adic context. The idea is therefore to characterize the icosahedron or any Platonic solid solely by its isometry group.

In practice this means following. Platonic solid is described as a collection of points. Vertices, midpoints of edges, and barycenters of faces. These points are fixed points for discrete subgroups of the Platonic solid. In the case of icosahedron the isometry group is A_5 the group of even permutations of 5 letters. There are are 6 cyclic subgroups of order 5, 10 cyclic subgroups of order 3, and 15 cyclic subgroups of order 2. The respective fixed points are the 12 vertices, 20 barycenters, and 30 midpoints of edges. Thus icosahedron becomes a collection of points with a label telling which is the cyclic subgroup associated with the point. This is something which might be able to generalize to p-adic context since there would be no need to talk about distances. One should however describe also the "solid" aspect of icosahedron.

2.2 What does one mean with ordinary 2-sphere?

In order to construct p-adic analog of icosahedron one must construct a space in which the isometry group A_5 of icosahedron acts and is imbedded to a group defining the analog of rotation group.

One could consider two options. The first option would be 3-D Euclidian space $E^3 \equiv R^3$ replaced with its p-adic counterpart Q_p^3. The action of SO(3) however leaves the distance from origin invariant and one can restrict the consideration to 2-sphere. The challenge is to define the counterpart of 2-sphere p-adically.

Before one can say anything about p-adic 2-sphere, one must understand what means with the ordinary 2-sphere identified now as sphere in metric sphere.

1. Riemann sphere is compactification of complex plane and can be regarded as complex projective space $CP_1 = P^1(C)$ is taken as starting point. This space is obtained from C^2 by identified points (z_1, z_2) which differ by a complex scaling: $(z_1, z_2) = \lambda(z_1, z_2)$. One can say that points of $P^1(C)$ are complex lines, which are nothing but Riemann spheres. This manifold requires two coordinate patches corresponding to patch containing North *resp.* South pole but not South *resp.* North pole. The coordinates in a patch containing Northern hemisphere can be taken to be $(u = z_1/z_2, 1)$ by projective equivalence allowing to select point $(z_1/z_2, 1)$ from the projective line with $z_2 \neq 0$. In the region containing Southern hemisphere one can take $v = z_2/z_1$). In the overlap region around equator the coordinates are related by $v = 1/u$. One can think also $P^1(C)$ as plane with single point ∞ (south pole) added.

2. The group $PGL(2, C)$ and also the Lorentz group $SL(2, C)$ acts at Riemann sphere as Möbius transformations. The complex matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is represented as a Möbius transformation

$$u \rightarrow \frac{au + b}{cu + d} \quad .$$

Note that the matrix elements are complex: what this means in p-adic context is not at all clear!

One can regard the coordinates z_1 and z_2 as spinor components and the action of SO(3) is lifted to the action of covering group SU(2) for which 2π rotation is represented by -1. The group A_5 can be lifted to its covering group have twice as many elements as the original one but the action of SU(2) *resp.* overing of A_5 reduces to that of SO(3) *resp.* A_5 since one considers the action on the ratio z_1/z_2 of the spinor components.

3. $S^2 = P^1(C)$ is a good structure to generalize to p-adic context since one can define it purely algebraically, and one realize the action of isometries in it.

2.3 Icosahedron in p-adic context

2.3.1 What does one mean with p-Adic numbers?

The article about p-icosahedron [A8] gives also a concise summary of p-adic numbers. p-Adic number fields define a hierarchy of number fields Q_p labeled by prime $p = 2, 3, 5, ...$ They are completions of rationals so that rationals can be said to be common to reals and p-adics. Each Q_p allows an infinite number of algebraic extensions whereas reals allow only one - complex numbers.

Local topology of p-adic numbers is what distinguishes them from reals. Two points of Q_p are near to each other if they differ by a very large positive power of p. As real numbers these numbers would differ very much. Most p-adic numbers have infinite number pinary digits in the pinary expansion and are infinite as real numbers.

The p-adic norm defining the p-adic topology is defined by p-adic number fixed completely by the lowest pinary digit in the expansion and is therefore very rough and obtains only values p^n for Q_p. The resulting topology is very rough. Indeed all p-adic points define open sets: one says that p-adic topology is totally disconnected. p-Adic norm is non-Archimedean. It satisfies $|x - y| \leq Max\{x, y\}$ whereas real norm satisfies $|x| - |y| \leq |x - y| \leq |x| + |y|$. This property of p-adic topology is known as ultrametricity.

p-Adic differential calculus exists and differentiation rules are same as for the real calculus. It is however not at all clear whether given real Taylor series with rational coefficients generalizes to its p-adic counterpart since the series need not converge p-adically. Exponential and trigonometric functions have p-adic counterparts but they do not have the properties of their real counterparts: for instance, p-adic trigonometric functions are not periodic. This is a problem when one tries to generalize Fourier analysis.

p-Adic integral calculus is problematic. The reason is that p-adic numbers are not well-ordered. As a consequence, the ordering crucial for Riemann integral does not exist. In fact, formal definition of Riemann integral gives as a limit vanishing integral. The generalization of Fourier analysis based on the integration of plane wave factors $exp(ikx)$ as roots of unity appearing in algebraic extension of p-adic numbers seems to be the only manner to overcome the problem. Algebraic continuation of integrals depending on parameters (such as integration limits) from real to p-adic context is in a central role in TGD framework but requires the fusion of reals and various p-adic number fields to bigger structure along common rationals: each number field would be like one page in a big book.

2.3.2 What does one mean with p-adic complex projective space?

The question is what one should do for the projective space $P^1(C)$ to get its p-adic counterpart? The basic condition is that A_5 acts transitively in the p-adic analog of $P^1(C)$.

1. The first guess would be the replacement of $P^1(C)$ with $P^1(Q_p)$. This is however the p-adic analog of real projective line, not complex projective line and one cannot imbed the complex matrices representing the action of the covering group of A_5 of $PGL(2, Q_p)$.

2. What one should do? The basic observation is that complex numbers C define the only possible algebraic extension of real numbers. Generalizing this, one should consider algebraic extension of Q_p. There is infinite number of these extensions and one must choose that of minimal algebraic dimensions. This means that the phases $exp(i\pi/5)$ (10:th root of unity), $exp(i\pi/3)$ (6:th root of unity), and $exp(i\pi/2) = i$ (4:th root of unity) must be contained by the extension. The reason why one must have $exp(i\pi/5)$ rather than $exp(2\pi/5)$ representing rotation of $2\pi/5$ generating the cyclic group Z_5 is due the fact that one has two fold covering. Same applies to other roots of unity. The solutions of equation $x^{60} = 1$ give the needed roots of unity since $60 = 6 \times 10 = 4 \times 3 \times 5$ contains all the needed roots of unity needed in the representation matrices.

   The extension of Q_p containing those roots of unity which do not reduce to -1 (existing p-adically) would define the extension used. One can calculate the algebraic dimension of this extension but certainly it is much larger than 2 as in the case of complex numbers. The extension - call it K - is not unique but is minimal. There is infinite number of extensions containing this extension.

To define things precisely one must replace the notions of p-adic integer, prime, and rational p applying in K but this is a technicality. This means that p - the only prime in Q_p - is replaced with π, the only prime in K.

I will leave the detailed construction of the projective space $P^1(Q_p)$ later because it is rather technical procedure. Some comments are however in order:

1. For $p \ mod \ 4 = 1$ (say $p = 5$ or 17) $i \equiv \sqrt{-1}$ belongs to the p-adic number field. Therefore the dimension of algebraic extension is considerably smaller than for $p \ mod \ 4 = 3$ (say $p = 3$ or 7) .

2. The naive question is whether for $p \ mod \ 4 = 3$ a considerably simpler approach could make sense. Use 2-D algebraic extension of p-adic numbers consisting of numbers $x+iy$: call this space C_p. Naive non-specialist might think that in this case the rather intricate complex construction of the projective space $P^1(Q_p)$ based on Bruhat-Tits tree might not be needed. This simpler construction however fails for $p \ mod \ 4 = 1$. It fails also more generally. The reason is that the $exp(i\pi/n)$, $n = 3, 5$ are algebraic numbers and do not belong to C_p. Therefore one must extend C_p to included also the phase factors and it seems that one ends up to the same situation as in general case.

3. Side track to TGD.

   (a) In TGD one encounters the problem "What could be the p-adic counterpart of S^2 and $CP_2 = P^2(C)$?". The above general recipe applies to this problem: replace C with an algebraic extension K of Q_p allowing the imbedding of some discrete subgroup of $SU(2)$ *resp.* $SU(3)$ represented as matrices in $PGL(2, K)$ *resp.* $PGL(3, K)$. The interpretation would be that due to finite measurement resolution the Lie group $SU(2)$ *resp.* $SU(3)$ is replaced with its discrete counterpart.

   This has a direct connection to the inclusions of hyperfinite factors of type II_1 (HFF) [K11], where all discrete subgroups of $SU(2)$ appear also those of SU(3), whose interpretation is in terms of finite measurement resolution with included HFF creating states which cannot be distinguished from the original state in the resolution used. General inclusions correspond to discrete subgroups of rotation group and by McKay correspondence [A12] to Lie groups of ADE type. The isometry groups of Platonic solids are the only simple groups in this hierarchy and correspond to exceptional Lie groups E_6, E_7, E_8.

   (b) One could criticize the approach since the algebraic extension K containing the isometry group is not unique. In TGD framework one however interprets the algebraic extensions in terms of finite measurement resolution. One cannot measure all possible angles p-adically-actually one cannot measure angles at all but only discrete set of phase factors coming as roots $exp(ik2\pi/n)$ of unity. The large the value of n, the better the measurement resolution.

2.3.3 What does one mean with p-adic icosahedron?

Once the projective space $P^1(K)$ generalizing $P^1(C) = S^2$ is constructed such that it allows the action of A_5 (it does not allow the action of entire rotation group!) one can identify the points which remain fixed by the action of various subgroups of A_5 (6 cyclic subgroups of order 5, 10 cyclic subgroups of order 3, and 15 cyclic subgroups of order 2. The respective fixed points are the 12 vertices, 20 barycenters, and 30 midpoints of edges). This is a purely algebraic procedure and there is no need to define what edges and faces are.

To obtain a more concrete picture about the situation one must define precisely what $P^1(Q)$ means and here the notion of Bruhat-Tits tree [A3] seems to be unavoidable.

3 Trying to explain what $P^1(Q_p)$ could mean technically

The naive approach to the construction of $P^1(Q_p)$ would be following. Do the same things as in the case of $P^1(C)$ or $P^1(R)$. The point pairs (q_1, q_2) in Q_p^2 are identified with pairs $\lambda \times (q_1, q_2)$ where $\lambda \neq 0$ is p-adic number. For some reason this simple approach is not adopted in the article [A8]. The reason is that one cannot introduce the notion of Bruhat-Tits tree [A3] in this approach. Bruhat-Trits tree is needed to obtain path-connectedness - that is connect the fixed points of icosahedron to form a "solid" and to give a more geometric meaning to the notion of icosahedron. One can regard $P^1(Q_p)$ as boundary of Bruhat-Tits tree somewhat like sphere is a boundary of ball in real context.

I am not not sure whether this approach on $P^1(Q_p)$ is equivalent with that of Berkovich [A14] based on the idea of adding some points to $P^1(Q_p)$ to make it path connected space containing $P^1(Q_p)$ as a dense subset. The outcome has rather frightening complexity.

The alternative approach would be purely algebraic. I will discuss later the problem of introducing the counterpart of path connectedness without giving up p-adic topology and by introducing induced real topology as effective topology having the desired path-connectedness.

3.1 Generalization of $P^1(C)$ making possible to introduce Bruhat-Tits tree

The following construction looks somewhat artificial but its purpose is to make possible the introduction of Bruhat-Tits tree allowing to realize path-connectedness.

1. The point pairs (q_1, q_2) Q_p^2 are replaced with Z_p lattices in Q_p^2. For given lattices the points are of form (n_1u, n_2v), where u and v are linearly independent (in Q_p) vectors of Q_p^2. Note that the p-adic integers $n_i = \sum_{k\geq 0} n_{i,k}p^k$ can be and typically are infinite as real integers. This is how the lattice differs from the real lattice. Also the p-adic distances between lattices points for which n_i differ by a large power of p are very small.

   Note: Q_p^2 is the p-adic analog of space of 2-spinors. The pairs (u, v) are indeed in 1-1 correspondence with pairs (q_1, q_2).

2. Projective equivalence is realized as for point pairs (q_1, q_2). This means that lattices for which base vectors (u, v) differ by a p-adic scaling are equivalent $(u, v) \equiv (\lambda u, \lambda v)$. Only the ratio u/v defining the "direction" of point of Q_p^2 matters.

   Note: In the complex case one would have two complex vectors and their ratio defines the conformal equivalence class of the plane compactified to torus by identifying the opposite edges of the polygon defined by u/v.

   Note: In the article one speaks about homothety classes: homothety means scaling which in p-adic context need not change p-adic norm.

This is not quite enough yet. Real icosahedron is in a well defined sense a connected coherent structure. Not just a collection of points. p-Adic topological is however totally disconnected. This suggests that one must introduce additional structure making possible to speak about icosahedron as "solid". Bruhat-Tits tree is one possible manner to achieve this. Also TGD inspired view about p-adic manifolds makes this possible.

3.2 Why Bruhat-Tits tree?

One introduces Bruhat-Tits tree [A3] as an additional structure having $P^1(Q_p)$ as its boundary in a well-defined sense (one needs its counterpart also in $P^1(K)$). In [A8] it is stated that this relates to a proper *global* definition of p-adic analytic structure in terms of Berkovich disks. As already explained, the basic problem for introducing analytic manifold structure is the total disconnectedness of p-adic topology. In p-adic topology each point is open set and all p-adic open sets are also compact. Moreover, two p-adic balls are either disjoint or nested. Therefore one cannot have partially overlapping p-adic spheres and the basic construction recipe for real manifolds fails. One can overcome this problem for algebraic varieties defined by algebraic equations but they are much less general objects than manifolds in real context.

1. There are no problems in defining p-adic differential calculus (a *local* aspect of the analytic structure) and field equations associated with action principles make sense although the definition of action as integral is problematic. p-Adic differential equations are non-deterministic: integration constants are replaced by piecewise constant functions depending on finite number of pinary digits. This has a nice interpretation in TGD inspired consciousness, where this non-determinism would be correlate for non-determinism of imagination - one aspect of cognition.

Therefore I am not at all sure whether the reinforcement of real number based notions to p-adic context is a good idea.

2. p-Adic integration (a *global* aspect of the analytic structure) is the problem in p-adic calculus and the total disconnectedness relates to the absence of well-ordering. An obvious guess is that Bruhat-Tits tree could help in the definition of p-adic integral by defining the allowed integration paths.

   Note: TGD approach on integration relies on algebraic continuation from real context and is based on what might ge regarded fusion of reals and p-adics along common rationals.

3. Intuitively the Bruhat-Tits tree builds up a "skeleton" connecting points by edges and thus curing the total disconnectedness. This requires some non-locality and the replacement of point pairs (q_1, q_2) with integer lattices spanned by q_1 and q_2 would introduce this non-locality.

4. In any case, what one obtains is a graph with vertices and edges. Vertices are identified as homothety classes $[M]$ of the lattices and are just the points of $P^1(Q_p)$. Two vertices $[M]$ and $[N]$ are connected by an edge iff one can find representatives M and N such that $pM \subset N \subset M$. The representative N is in some sense between pM and M. Note that one has $pM \equiv M$ by homothety so that the use of representatives in the definition is necessary.

   The resulting graph is also a regular $p+1$-valent tree, the number of F_p-rational points of $P^1(F_p)$,which is projective space associated with finite field. One can check this in case of $p = 2$. The points (f_1, f_2) are $(1, 0), (1, 1), (0, 1), (1, 1)$ and by projective equivalence one has just $p = 1 + 2 = 3$ points in corresponding projective space. The transitive action of $Gl(2, K)$ means that all vertices are $p + 1$-valent and this fixes the structure of the graph completely. I will consider this point in more detail later on basis of the web article [A3].

Bruhat-Tits tree can be seen as a skeleton of the "full" $P^1(K)$ containing also the additional points making it a path connected Berkovich space. The "naive" $P^1(K)$ can be regarded as boundary of the Bruhat-Tits tree.

Bruhat-Tits tree looks very nice notion but there is objection against its construction in the proposed manner. Ordinary p-adic numbers- the simplest possible situation - are not in 1-1 correspondence with the Z_p lattices as will be demonstrated later but with powers of p. Same applies to Q_p^2 where the lattices correspond to $Sl(2, Z_p)$ equivalence classes of elements of Q_p^2. One can of course ask whether projective spaces are p-adically and maybe also physically very special for this reason.

3.3 Berkovich disk

Bruhat-Tits tree is not enough for p-adicizing real topologist. Also Berkovic disk is required as the analog of open ball in real context. The slides of Emmy Noether Lecture by Annette Werner [A14] give a concise representation of the basic idea behind Berkovich disk serving as a basic building brick of p-adic manifolds just like real n-disk does in the case of real n-manifolds and also also explains its construction. I must admit that I do not understand well enough the connection between Berkovich disk and Bruhat-Tits tree.

One can motivate the construction with the completion of rationals to reals. By adding all irrationals (algebraic numbers and transcendentals) one obtains reals and these additional numbers glue the rationals to form a continuum so that one can defined calculus and many other nice things. The idea is to mimic this construction.

1. In the example one restricts to the unit disk for an non-archimedean field assumed for simplicity be algebraically closed, which means algebraic completion containing all algebraic numbers considered also by Khrennikov. This notion is very formal and unpractical. The idea is to form a completion of the unit disk for a non-archimedean field K (algebraic extension of Q_p) containing thus K as a dense subset with the property that the resulting topology is path connected and not anymore ultrametric (somewhat artificial!).

   For this purpose one constructs what is called the space of bounded multiplicative non-Archimedean norms for formal K-valued power series defined in the unit disk reducing to the norm of K for constant functions. It is possible to characterize rather explicitly this space and with topology defined by a pointwise convergence (point is now the K-valued function) of the norm one obtains uniquely path connected topology. The additional points can be said to glue the points of the K-disk to a continuum as its dense subset just as the addition of irrationals glues rationals to form a continuum.

2. The construction generalizes to the construction of the counterparts of p-adic projective spaces and symmetric spaces. Berkovich has also proposed an approach to p-adic integration and harmonic analysis relying on the notion of Berkovic space.

   Note: In TGD framework integration is defined by algebraic continuation in the structure defined by the fusion of real and various p-adic numbers fields and their extensions to form a book like structure. One could perhaps say that this fusion defines a kind of "super-completion": all possible completions of rationals are fused to single book like structure and rationals indeed defined a dense subset of this structure.

The construction is rather technical. From unit disk to a function space defined in it to the space of multiplicative seminorms defined in this function space! For the simple brain of physicist desperately crying for some concreteness this looks hopelessly complicated. Physicists would be happy in finding some concrete physical interpretation for all this.

3.4 Bruhat-Tits tree allows to "connect" the points of p-adic icosahedron as a point set of $P^1(K)$

The notion of p-adic icosahedron can be defined also in terms of Bruhat-Tits tree since the $PLG(2,K)$ acts transitively on the homothety class so that one obtains all homothety classes from the one associated with $(u,v) = (1,1)$ and one can speak about orbit of this basic homothety class. This means that one can connect the vertices, mid-points of edges, and barycenters of faces to common origin by edge paths in Bruhat-Tits tree and therefore to each other. This is what path-connectedness means.

How Bruhat-Tits tree allows to build from a set of totally disconnected fixed points a "solid"? One answer is that the addition points of completion make this possible.

1. Bruhat-Tits tree allows to define what is called an end of the Bruhat-Tits tree as an equivalence class of infinite half line with two half lines identified if they differ by a finite number of edges. These ends are in one-one corresponds with the K-rational points of $P^1(K)$ (these are not the only points of $P^1(K)$). One can say that $P^1(K)$ represents the boundary of Bruhat-Tits tree as a p-adic manifold.

   Note: Could this finite number of different edges corresponds to a finite number of pinary digits appearing in p-adic integration "constants")? The identification could mean that all choices of pseudo constants in p-adic differential equations are regarded as equivalent. Physicist might speak about the analog of gauge invariance: the values of pseudoconstants do not matter.

2. For a finite set of points of totally disconnectd $P^1(K)$ there exists a unique minimal subtree of the entire Bruhat-Tits tree containing the points of this set as its ends [A8]. This subtree is what connects the points of this point set to a coherent structure in the set that one can construct paths connecting the points to single point. There are of course several manners to achieve this but one can define even the analog of the geodesic line as a path with a minimal number of edges so that it becomes possible to speak also about the edges of icosahedron. The length of the geodesic could be simply the number of edges for this minimal edge path.

3. The p-adic counterpart of Platonic solid must be also "solid". This is achieved if the fixed points for the subgroups of the isometry group of Platonic solid (in particular for those of the A_5) defining the Platonic are identified as ends of a unique minimal subtree of Bruhat-Tits tree.

For higher-dimensional projective spaces $P^n(K)$ Bruhat-Tits tree generalizes from 1-D discrete homogenous space $PGl(2,K)/Gl2,Z_K)$ to n-dimensional discrete homogenous space. The reason is that the edges of tree develop higher-dimensional cycles having interpretation as simplexes. One can also define homology groups for this structure. Also now $P^n(K)$ can be regarded as a boundary of the resulting structure.

4 Algebraic universality in TGD framework

In TGD framework the algebraic approach looks very promising one - at the first glance perhaps even the only possible one - since the field equations for preferred extremals [K1, K13] reduce to purely algebraic ones and do not even refer to action principle explicitly. The point is that the preferred extremal property means a generalization of complex structure to 4-D situation and is a notion independent of action and the preferred extremals are solutions to field equations of very many general coordinate invariant variational principles (Einstein-Maxwell equations with cosmological term and minimal surface equations hold true). p-Adic variants of these conditions are purely algebraic and make sense so that one can hope that even space-time surfaces might have p-adic counterparts.

As already noticed, one can consider a compromise between topological and algebraic approach to the definition of p-adic manifolds by using a variant of canonical identification to map rational points of the p-adic preferred extremal to rational points of its real counterpart and completing this skeleton to a preferred extremal in the real context. This mapping need not be one-to-one. In the intersection of real and p-adic worlds the expression for real preferred extremal makes sense also in p-adic number field, and a direct identification makes sense and is unique.

In the real sector the preferred extremal property would boil down to to the existence of complex structure in Euclidian regions and what I call Hamilton-Jacobi structure in Minkowskian regions. Also the conjecture that preferred extremals are quaternionic surfaces in certain sense [K10] implies independence on action principle. The challenge is to prove that these two algebraic characterizations of preferred extremals are equivalent. These two purely algebraic conditions might make sense also in p-adic context with complex and hypercomplex numbers replaced with appropriate algebraic extensions of p-adic numbers.

The p-adicization program based on the notion of algebraic continuation involves many open questions to be discussed first.

4.1 Should one p-adicize entire space-time surfaces or restrict the p-adicization to partonic 2-surfaces and boundaries of string world sheets?

One of the many open questions concerns the objects for which one should be able to find p-adic counterparts. The arguments based on canonical identification and universality of the preferred extremal property support the view that p-adicization can be carried out at 4-D level for space-time surfaces and also at the level of WCW. Later a detailed proposal for how p-adic preferred extremals can be mapped to real preferred extremals with the uniqueness of this correspondence restricted by the finite measurement resolution realized as pinary cutoff will be described.

One can however consider also an alternative approach in which one restricts the p-adicization to 3- or even 2-dimensional objects of some special classes of these objects and this possibility is discussed below.

1. Should one p-adicize only boundaries?

A grave objection against p-adicizing only partonic 2-surfaces and braid strands is that one loses the very powerful constraints provided by the preferred extremal property and coordinate maps defined by the canonical identification in preferred coordinates. Therefore the algebraic continuation of the partonic 2-surface can become highly non-unique ($x^n+y^n=z^n$, $n>2$, is the basic counter example: in higher dimensions one expects that this kind of situations are very rare!). Furthermore, the restriction to partonic 2-surfaces and braid strands is artificial since imbedding space must be p-adicized in any case. The replacement of the p-adicization of the partonic surface plus 4-D tangent space data with that of the preferred extremal containing it increases the number of constraints dramatically so that holography might even make the p-adicization unique.

Despite this objection one can try to invent arguments for restricting the p-adicization to some subset of objects since this would simplify the situation enormously.

1. The basic underlying idea of homology theory is that the boundary of a boundary is empty. p-Adic manifolds in turn have no boundaries because of the properties of p-adic topology. Should p-adicization in TGD framework be carried only for boundaries? Light-like 3-surfaces define boundaries between Minkowskian and Euclidian regions of space-time surface. The space-like 3-surfaces defining the ends of space-time surfaces at the boundaries of CD are boundaries.

Also 2-D partonic surfaces and boundaries of string world sheets can be considered. One must consider also the boundaries of string world sheets as this kind of objects.

2. Strong form of General Coordinate Invariance implies strong form of holography. Either the data at light-like 3-surfaces (at which the signature of induced metric changes) or space-like 3-surfaces at the ends of CD codes for physics, which implies that partonic 2-surfaces and 4-D tangent space data at them code for physics.

   What 2-D tangent space data could include? The tangent space data are dictated partially by the weak form of electric magnetic duality [K4] stating that the electric component of the induced Kähler field component is proportional to its magnetic component at light-like 3-surfaces. Also the boundaries of string world sheets contribute to 4-D tangent space data and at the end of braid strands at partonic 2-surfaces both light-like and space-like direction are involved.

If space-time interior is not p-adicized (somewhat un-natural option), the p-adicization reduces to the algebraic continuation of Kähler function and Morse function to p-adic sectors of WCW. Both functions reduce to 3-D Chern-Simons terms for selected 3-surfaces. p-Adicization should reduce to algebraic continuation of various geometric parameters appearing as arguments of Kähler action.

In the minimal situation only partonic 2-surfaces and the boundaries of string world sheets - briefly braid strands - need to be p-adicized and the existing results - such as the results of Mumford derived from the existence of p-adic uniformization - could give powerful contraints. One can also ask whether the p-adic string world sheet in some sense is equivalent with the generalization of Bruhat-Tits tree allowing also loops.

Besides the string world sheet boundary and partonic 2-surface also for "4-D tangent space data" fixed at least partially by weak form of electric magnetic duality and string world sheets is needed. There are several open questions.

1. Does weak form of electric-magnetic duality have any meaning if one cannot speak about space-time interior in p-adic sense? This condition would apply only at partonic 2-surfaces. Same question applies in the case of braid strands. Can one effectively reduce space-time interior and string world sheet to their tangent spaces at partonic 2-surface/braid strands.

2. It is not even clear whether the dynamics of light-like 3-surfaces and space-like 3-surfaces is deterministic. Strong form of holography requires either determinism or non-determinism realized as gauge invariance, which could correspond to Kac-Moody type symmetries. Kac-Moody symmetry would favor the idea that p-adicization takes place only for partonic 2-surfaces and for the braid strands. Gauge symmetry would also give hopes that the integral of Chern-Simons term depends only on the data at the end points of braid strands at partonic 2-surfaces and maybe on data at braid strands: this would however require p-adic integration not possible in purely p-adic context. These data should remain invariant under Kac-Moody symmetries.

3. Should one p-adicize the weak form of electric magnetic duality? The duality involves the dual of Kähler form of the partonic surface with respect to the induced four-metric: the normal component of Kähler electric field at partonic surface and/or at string world sheet boundary equals to Kähler magnetic form at the partonic surface at particular point of its orbit (most naturally light-like curve). The induced 4-metric becomes degenerate at the light-like 4-surface and the component of electric field is finite only if weak form of electric-magnetic duality can be satisfied. Should the duality hold true for entire 3-surfaces, for partonic 2-surfaces, or perhaps only for for the braid strands? The purpose of the condition is to guarantee that Kähler electric charge as eletric flux is proportional to Kähler magnetic charge: therefore it should hold along entire 3-surfaces and if these are regarded as real surfaces there are no problems with the p-adicization of the condition.

2. What kind of algebraic 2-surfaces can have p-adic counterparts?

There is no need for a generic algebraic surface to have direct algebric p-adic counterpart for all p-adic primes. If one uses as preferred coordinates a subset of preferred coordinates of the imbedding space and accepts only imbedding space isometries as general coordinate transformations, the algebraic surfaces in the intersection of real and p-adic worlds must satisfy very strong conditions. For instance, a representation in terms of polynomials cannot involve real transcendentals. Even rational coefficients can force algebraic extension of Q_p, when the remaining imbedding space coordinates are expressed in terms of the coordinates of the partonic two-surface.

Mumford is one of the pioneers of p-adicization of the algebraic geometry and has demonstrated that only a restricted set of p-adic algebraic surfaces allow interpretation as p-adic Riemann surfaces if one requires that a generalization of so called uniformization theorem holds true for them [A5]. This theorem says that Riemann surfaces are constructible as factor spaces of either sphere, complex plane, or complex upper plane (hyperbolic space H^2 with the subgroup Γ identified as the finitely generated free subgroup of the isometries of the space in question. The construction does not work for all algebraic surfaces but only for the surfaces satisfying certain additional conditions. This is not a problem in TGD framework in the intersection of real and p-adic worlds since the p-adicization is not expected to be possible always but only in the intersection of real and p-adic worlds.

According to the article Multiloop Calculations in p-Adic String Theory and Bruhat-Tits Trees by Chekhov et al [A9] the construction of higher genus Riemann surfaces as so called Mumford surfaces takes place by starting from Bruhat-Tits tree representing $g = 0$ surface and by taking subgraphs having interpretation as representations for an orbit of so called Schottky group characterizing the higher genus Riemann surface and gluing these graphs together by transversal connections. This indeed represents the genus homologically as a loop of the resulting tree.

Note: The article of Chekhov et al describes a proposal for the construction of complex scattering amplitudes for p-adic strings in real imbedding space so that the situation is not relevant for TGD as such. The amplitudes are constructed in terms of p-adic characteristics and this means that the amplitudes can be interpreted also as numbers in p-adic number fields extended by roots of unity. The characteristics $q = exp(i2pi\tau)$ exist only for the values of q which are of form $q = p^n exp(x)exp(i2\pi/m)$, $|x| < 1$ so that discretization of the p-adic norm and phase of τ is necessary.

3. Should one really restrict the p-adicization to algebraic surfaces?

One could also consider the possibility of restricting p-adicization to algebraic surfaces (they could be also 4-D). Practicing physicist would argue that the restriction of p-adicization to algebraic surfaces is quite too heavy an idealization. In the real world spheres are topological rather than algebraic.

Luckily, if the construction recipe for p-adic manifolds to be discussed later really works, canonical identification with pinary cutoff allows to generalize p-adic algebraic surfaces to p-adic manifolds, and to achieve very close correspondence with the real manifold theory. Given real preferred extremal can correspond to not necessarily unique p-adic preferred extremal for some values of p. Also two p-adic preferred extremals with different values of p-adic prime which correspond to the same real preferred extremal correspond to each other. This provides an elegant solution to all problems discussed hitherto and there is not need to restrict the p-adicization in any manner.

Finite measurement resolution would be a prerequisite for algebraic continuation in the sense that subset of rational and algebraic points defined by pinary cutoff and algebraic extension would be common to the real and p-adic preferred extremals. Therefore finite measurement resolution would make it possible to realize both number theoretical universality and p-adic manifold topology.

4.2 Should one p-adicize at the level of WCW?

One can of course challenge the idea about p-adicization at the level of WCW and WCW spinor fields and ask what this procedure gives. One motivation for the p-adicization would be p-adic thermodynamics. p-Adic thermodynamics should emerge at the level of M-matrix which indeed can be regarded as a "complex square root" of hermitian density matrix in zero energy ontology and therefore expressible as a product of hermitian square root of density matrix and unitary S-matrix. Hence it would seem that the p-adicization at the level of WCW is natural and the representability as a union of symmetric spaces constructible as factor groups of symplectic group of $\delta M^4_\pm \times CP_2$ gives hopes that algebraic approach works also in infinite-dimensional case. Finite measurement resolution and the properties of hyper-finite factors of type II_1 are expecetd to reduce the situation to finite-dimensional case effectively.

4.3 Possible problems of p-adicization

The best manner to clarify one's thoughts is to invent all possible objections and in the following I do my best in this respect. The basic point is following. If one accepts the purely algebraic approach without no reference to canonical identification, one must check that everything in TGD - as I recently understand it - can be expressed without inequalities! Boundaries are defined by inequalities and one must check that they can be avoided. If this is not the case, the notion of p-adic manifold relying on the notion of canonical identification seems to remain the only manner to avoid problems.

4.3.1 Wormhole throats are causal rather than topological boundaries

The notion of boundary does not have any counterpart in purely p-adic context since its definition involves inequalities. The original vision was that space-time sheets possess boundaries and the boundaries carry quantum numbers - in particular family replication phenomenon for fermions would have explanation in terms of the genus of 2-dimensional boundary component of 3-surface [K3]. It however turned out that boundary conditions require that the space-time sheet approaches vacuum extremals at boundary and this does not seem to make sense. This led to the view that one must allow only closed space-time "sheets" which can be thought of as being obtained by gluing real space-time sheets together along boundaries.

Also the notion of elementary particle involves preferred extremals - massless extremals in the simplified model [?] connected by wormhole contact structure defining the elementary particle. These preferred extremals must combine to form a closed space-time surface and this is quite possible: the minimal situation corresponds to two space-time sheets glued together as in the model of elementary particles.

Genuine boundaries are replaced by the light-like 3-surfaces -orbits of wormhole throats - at which the signature of the induced metric changes from Minkowskian to Euclidian and four-metric degenerates effectively to 3-D metric locally. These can be defined by purely algebraic conditions and there is no need for inequalities.

Partonic 2-surfaces are identified as intersections of the space-like 3-surfaces at the ends of CD: the ends of CD are defined by purely algebraic equation $t^2 - r^2 = 0$ and $(t - T)^2 - r^2 = 0$ and once the equations of space-time surface are known one can solve the equations for space-like 3-surfaces. The equations defining what light-like 3-surfaces at which the induced four-metric is degenerate are algebraic and express just the degeneracy of the induced four-metric. The condition that algebraic equations for light-like 3-surfaces and space-like 3-surfaces hold true simultaneously define partonic 2-surfaces. Hence it seems that the surfaces can be expressed algebraically.

This approach might look a little bit artificial. Also the idea that only boundaries should be p-adicized should be be p-adicized looks artificial. The best looking option is the use of canonical identification to define p-adic manifolds since it allows to transfer real topological notions to the p-adic context. In particular, the well-ordering of reals induces that of p-adics so that inequalities cease to be a problem and boundaries can be defined.

4.3.2 What about the notion of causal diamond and Minkowski causality?

A possible problem for purely p-adic approach allowing no in-equalities is caused by the notion of causal diamond (CD) defined as intersection of future and past directed light cones (as a matter fact, CP_2 is included to CD as Cartesian factor but I do not bother to mention it again and again). CD has light-like boundaries.

It is not quite clear whether space-time surface must be always localized inside CD. The notion of generalized Feynman diagram indeed suggests that the space-time surfaces can continue also outside the CDs and that CD could be seen as an imbedding space correlate for what might be called spot-light of consciousness. If this were the case quite generally, the p-adicization of space-time sheets would not produce problems even if one does not use canonical identification.

In purely p-adic context, one should however give some meaning for the statement that space-time surface is contained inside CD and this seems to require the notion of boundary for CD. Does this notion of CD make sense in the p-adic context or is the fusion of real and p-adic number fields along common rationals required? The resolution of the problem seems to require the fusion. In the case of algebraic extensions also common algebraics are present.

The first questions concern the notion of Minkowski causality, which relies on light-cone and its complement expressed in terms of inequalities.

1. The first reason of worry is that in purely p-adic context also the equation $t^2 + r^2 = 0$ has a lot of solutions! The reason is that the notion of positive and negative do not make sense for p-adic numbers without some constraints. If one restricts the p-adic numbers to those having finite number of pinary digits - this happens always when one has finite pinary resolution - all p-adic numbers included rationals reduces to finite positive integers as real numbers. Therefore in finite pinary resolution the problems disappear. The condition that rationals points of Minkowski space are common with its p-adic variant, makes finite pinary resolution natural, and one could say that all p-adic numbers - including negatives of finite integers - can be said to be infinitely large positive integers in real sense. Here one must of course be very cautious.

2. The condition $s = t^2 - r^2 < 0$ for the complement of future light-cone has no meaning in the p-adic context for general p-adic numbers. If rational values of Minkowski coordinates correspond to same point in real and p-adic sense, finite pinary resolution means that all pinary cutoffs have $s \geq 0$ and $t \geq 0$ in real sense. This is also true for $a = \sqrt{t^2 - r^2}$ so that one remains inside future light-cone unavoidably. Anything outside future light-cone is unexpressible in finite measurement resolution p-adically.

   Finite temporal and spatial resolution suggest integer quantization of t and r in suitable units and one could say whether s has finite of infinite number of pinary digits - that is are positive or negative. Finite real integer values of t and r have finite number of pinary digits. Their negatives have infinite number of pinary digits and one could argue they correspond to infinite future if they are interpreted as real numbers. The values of s in future light-cone have finite number of pinary digits and correspond to finite real values. Outsider the future light cone the values of s are negative in real sense and have infinite number of pinary digits and thus interpreted as real numbers are in future infinity.

   One can consider also rational values of t and s if one keeps also p-adically track that rational is in question. Rationality means that pinary expansion is periodic after some pinary digit. Therefore it would seem to be possible to distinguish between $s \geq 0$ and $s \leq 0$ also p-adically for finite measurement resolution purely algebraically.

3. Causal diamond is defined as the intersection of future and past directed light cones. The lower light-cone in the intersection decomposes to pieces of hyperplanes $t \geq 0$ with $r \leq t$ and upper light-cone to pieces $T - t \geq 0$, $r \leq T - t$. If these variables are quantized as integer multiples of suitable unit and if these integer multiples can be interpreted in both real and p-adic sense, there is no need for inequalities in p-adic context. Also now rational values can be allowed.

If only boundaries are p-adicized, p-adicization would apply only to the light-like boundaries of CDs, and one would avoid possible problems related the sign of $s = t^2 - r^2$. This would conform with the strong form of holography and allow p-adicization of WCW.

Again one might argue that the number theoretical game above is artificial. The safest alternative seems to be canonical identification with pinary cutoff used to map real preferred extremal to its p-adic counterpart.

4.3.3 Definition of integrals as the basic technical problem

Physicist wants to perform integrals, and the problems related to the notion of integral is what any novice of p-adic physics is doomed to encounter sooner or later. As will be described the definition of p-adic manifold based on canonical identification solves these problems by inducing real integration to the p-adic realm by algebraic continuation.

Before continuing about integration it is however good to summarize the general TGD based view about the relationship between real and p-adic worlds.

1. Intersection of real and p-adic worlds as key concept

In TGD framework the basic notion is the intersection of real and p-adic worlds generalizing the idea that rationals are common to reals and p-adics. Algebraic continuation between real and p-adic worlds takes place through this intersection, in which real formulas allow interpretation as p-adic ones. The notions of intersection and algebraic continuation apply both at space-time level and WCW level.

1. At the space-time level rational (and even some algebraic) points of real surfaces are contained by p-adic surfaces. One can identify these rationals and say that real and p-adic surfaces intersect at these points and define discrete cognitive representation. Among other things this would explain why numerics is necessarily discrete and possible only using rationals with cutoff.

2. One can abstract this idea to the level of WCW. Instead of number fields one considers surfaces (partonic 2-surfaces, 3-surfaces, or space-time surfaces) in various number fields. If the representation of the surface (say in terms of rational functions) makes sense both for reals and p-adic number field in question, one can identify the real and p-adic variants of surfaces. These surfaces can be said to belong to the intersection of real and p-adic worlds (worlds of classical worlds, to be more precise). In TGD inspired theory of consciousness one would say that they belong to the intersection of material/sensory world and the world of cognition. In TGD inspired quantum biology life is identified as something residing in the intersection of realities and p-adicities.

2. Algebraic continuation as a basic tool

With this philosophical background one an consider the algebraic continuation of real integrals from the intersection of real and p-adic worlds defined by surfaces, whose representations in preferred coordinates make sense in real number field and in the p-adic number field to which one wants to continue.

1. Harmonic analysis in coset spaces with discretization defined by the algebraic extension of Q_p might make possible to avoid the problems by reducing the integrals to sums over the discrete points of the coset space. Algebraic continuation is of course central element in the program.

2. The recent progress in the calculation of planar scattering amplitudes in $\mathcal{N} = 4$ SYMs gives hopes that M-matrix could be defined in number theoretically universal manner. The reason is that in TGD framework the fermions defining building bricks of elementary particles are massless - a basic prerequisite for the twistor approach - also when they appear as virtual particles. This gives enormously powerful kinematical constraints reducing the number of diagrams dramatically, and allows to express amplitude in terms of on-mass shell amplitudes just as one does in the twistor Grassmannian approach.

   For $\mathcal{N} = 4$ SYM (and also more general theories) planar Feynman diagrams boil down to integrals over Grassmannians, which are coset spaces associated with $Gl(n, C)/Gl(n - m, C) \times Gl(m, C)$ allowing the already described generalization to p-adic context. The integrals reduce to multiple residue integrals, which could make sense also in the p-adic context because of the very weak dependence on integration region. The algebraic continuation of the resulting amplitudes to p-adic context replacing C with an appropriate extension of p-adic numbers might well make sense.

3. Two problems as solutions of each other

Unfortunately, the algebraic continuation of integrals is not free of technical problems. Even in the case of rational functions the algebraic continuation of the real integrals is susceptible to p-adic existence problems.

1. The basic problem with definition of ordinary 1-D integrals of rational functions is that the integral function of $1/x$ is $log(x)$ rather than rational function as for other powers. Unless the limits are very special (of form $x = 1 + O(p)$), the algebraic continuation requires infinite-dimensional extension of p-adic numbers containing all powers of $log(x)$ for some $1 \leq x < p$. Can one allow infinite-D extensions, which are not algebraic?

2. The appearance of 2π in residue integral formulas which could otherwise make sense in p-adic context provides a second reason for worries: should one also transcendental extension containing powers of 2π?

Often two quite unrelated looking problems turn out to have a common solution. Now the second problem is purely physical: why a given particle should correspond to a particular p-adic prime? At this moment one must be satisfied with the p-adic length scale hypothesis stating that these primes are near powers of 2 and Mersenne primes are favored. I have not been able to identify any convincing dynamical principle explaining why primes near powers of two seem to be favored. It deserves however to be mentioned that the preferred p-adic length scale as a fixed point of p-adic coupling constant evolution (discrete) is one possible explanation meaning vanishing of beta functions, something very natural taking into account the quantum criticality of TGD Universe.

Could this problem define the solution of the first problem and vice versa! Maybe one must just accept that algebraic continuation to given p-adic number field is not always possible!

1. This criterion could strongly constrain the p-adic primes assignable to a given elementary particle. Consider as an example Kähler function defined as Kähler action for Euclidian portion of space-time (generalized Feynman graph) and Morse function defined as Kähler action for Minkowskian portion of space-time. The existence of the p-adic variant of Kähler function (or its real exponent) and Morse function (or its imaginary exponent) would allow to assign to a given space-time surface a highly restricted set of p-adic primes, and the allowed quantum superpositions of space-time surfaces could contain only those for which at least one of the allowed primes is same.

2. For massless particles Kähler action would vanish and algebraic continuation of Kähler action would be possible to all p-adic primes in accordance with the scale invariance of massless particles. Also the breaking of scale invariance and conformal invariance meaning selection of a particular p-adic length scale could be basically a number theoretical phenomenon. This would provide a totally new approach to the mystery of mass scales which in standard model framework requires fine tuning of Higgs mass with a totally unrealistic accuracy (one must avoid both the Landau pole meaning infinite self-coupling of Higgs and vacuum instability preventing massivation by Higgs vacuum expectation).

3. For instance, a function of form $log(m/n)$ can be algebraically continued only to those p-adic number fields for which m and n have form $m = k + O(p)$, and $n = k + O(p)$, $0 < k < p$ so that one has $m/n = 1 + O(p)$. The exponent of Kähler function in turn can be continued to Q_p if it is proportional to power of corresponding prime p. The exponential decay of Kähler function would have p-adic counterpart as decay of p-adic norm (just like Boltzmann weight $exp(-E/T)$ corresponds to p^n in thermodynamics). This could partially answer the question why the space-time surfaces assignable to electron seem correspond to Mersenne prime $M_{127} = 2^{127} - 1$ as suggested by p-adic mass calculations.

4. Number theoretic criterion might also mean that the p-adic prime characterizing particle state is extremely sensitive to the details of the particle state in real sense. The point is that a small modification of rational number in real sense changes its prime decomposition dramatically! Number theoretic anatomy is not continuous in real sense! An extremely small symmetry breaking in real sense modifying the value of Kähler function as function of quantum numbers might modify the value of the p-adic prime dramatically by affecting profoundly the number theoretic anatomy of some rational parameter appearing in the formula for Kähler function. For instance, in the standard framework it is very difficult to imagine any breaking for the SUSY assignable to right-handed neutrinos since they interact only gravitationally. The addition of right handed neutrino transforming particle to sparticle might however modify the p-adic prime (and thus mass scale) assigned to the particle dramatically.

4. What should one achieve?

It is a long way from this heuristic number theoretic vision to the calculation of p-adic valued integrals at space-time level, say to a formula for the p-adic action integral defined by Kähler action density (if needed at all).

1. The reduction to integral of Abelian Chern-Simons form over preferred 3-surfaces would be the first step and the definition of p-adic integral of Chern-Simons form second step. The special properties of preferred extremals give hopes about the reduction of the value of the

Kähler action to local data given at discrete points at partonic 2-surfaces. The braid picture for many-fermion states forced by the modified Dirac equation [K13] and motivated by the notion of finite measurement resolution having discretization as a space-time correlate, suggests a reduction of real action integral to a sum of contributions from the ends of braid strands defining the boundaries of string world sheets. The optimistic hope would be that this data allows a continuation to the p-adic realm.

Note: This kind of reduction might be quite too strong a conditon. All that is required in the approach based on canonical identification is that the values of Kähler function and Morse function exist in the given p-adic number field or its algebraic extension.

2. p-Adic valued functional integral is unavoidable at the level of WCW.

   (a) Algebraic continuation in the framework provided by the fusion of reals and various p-adic number fields looks the only reasonable approach to the p-adic functional integral.

   (b) Second element is Fourier/harmonic analysis in symmetric spaces: WCW is indeed a union of infinite-dimensional symmetric spaces over zero modes which do not contribute to WCW metric. One can hope that one can define the symmetric spaces algebraically in terms of their maximal symmetries since the metric reduces to that in single point of the symmetric space.

   (c) Canonical identification is the third element: p-adic functional integral for given p should be real functional integral restricted to preferred extremals allowing canonical identification map to the p-adic preferred extremal for that value of p. This would mean that real functional integral decomposes into a sum of contributions labelled by p-adic number fields and their algebraic extensions. This decomposition would be analogous to the formula obtained as a logarithm of the adelic formula for the rational as the inverse of the product of its p-adic norms.

4.3.4 Do the topological invariants of real topology make sense in the p-adic context?

In p-adic context the direct construction of topological invariants is not possible. For instance, the homology theory formulated in terms of simplexes fails since the very notion of simplex requires inequalities and well-ordering of the number system to define orientation for the simplex.

Also the notion of boundary is lacking since p-adic sets do not possess boundaries in topological sense. There however exists refined theories of p-adic homology allowing to circumvent this difficulty and the problem is that there are too many theories of this kind. A single universal theory would be needed and this was the dream of Grothendieck.

p-Adic mass calculations assume that the genus of the partonic 2-surface makes sense also in the p-adic context. For algebraic varieties the genus can be defined algebraically. There should be no problems if the partonic 2-surfaces are defined by algebraic equations which make sense for both reals and p-adic numbers. This is true for polynomial equations with rational coefficients and for algebraic extensions with coefficients in algebraic extension. By continuity algebraic continuation should allow to extend the notion of genus to surfaces for which rational coefficients are replaced with general p-adic numbers.

One expects that also more refined topological invariants making sense in the real context make sense also p-adically for algebraic varieties. A possible objection is that in the case of 3-manifolds allowing hyperbolic geometry (constant sectional curvatures) the volume of 3-manifold serves as a topological invariant. Volume is defined as an integral but in purely p-adic context volume integral is ill-defined. Is this a reason for worries? Hyperbolic n-manifolds have purely group theoretic formulation as coset spaces H^n/Γ, where Γ is discrete subgroup of the isometry group $SO(1,n)$ of n-dimensional hyperboloid H^n of $n+1$-D Minkowski space satisfying some additional conditions. Maybe this could allow to overcome the problem.

If canonical identification is used to map real preferred extremals to p-adic ones, boundaries and real topological invariants are mapped to p-adic ones both by algebraic continuation and in topological sense within finite measurement resolution. This even in the case that the real surface is not algebraic surfaces. This applies also to conformal moduli of the partonic 2-surfaces, whose p-adic variants play a key role in p-adic mass calculations.

4.3.5 What about p-adic symmetries?

A further objection relates to symmetries. It has become already clear that discrete subgroups of Lie-groups of symmetries cannot be realized p-adically without introducing algebraic extensions of p-adics making it possible to represent the p-adic counterparts of real group elements. Therefore symmetry breaking is unavoidable in p-adic context: one can speak only about realization of discrete sub-groups for the direct generalizations of real symmetry groups. The interpretation for the symmetry breaking is in terms of discretization serving as a correlate for finite measurement resolution reflecting itself also at the level of symmetries.

1. Definition of p-adic Lie groups

The above observation has led to TGD inspired proposal for the realization of the p-adic counterparts symmetric spaces resembling the construction of $P^1(K)$ in many respects but also differing from it.

1. For TGD option one considers a discrete subgroup G_0 of the isometry group G making sense both in real context and for extension of p-adic numbers. One combines G_0 with a p-adic counterpart of Lie group G_p obtained by exponentiating the Lie algebra by using p-adic parameters t_i in the exponentiation $exp(t_iT_i)$.

2. One obtains actually an inclusion hierarchy of p-adic Lie groups. The levels of the hierarchy are labelled by the maximum p-adic norms $|t_i|_p = p^{-n_i}$, $n_i \geq 1$ and in the special case $n_i = n$ - strongly suggested by group invariance - one can write $G_{p,1} \supset G_{p,2} \subset ...G_{p,n}....$ $G_{p,i}$ defines the p-adic counterpart of the continuous group which gets the smaller the larger the value of n is. The discrete group cannot be obtained as a p-adic exponential (although it can be obtained as real exponential), and one can say that group decomposes to a union of disconnected parts corresponding to the products of discrete group elements with $G_{p,n}$.

   This decomposition to totally uncorrelated disjoint parts is of course worrying from the point of view of algebraic continuation. The construction of p-adic manifolds by using canonical identification to define coordinate charts as real ones allows a correspondence between p-adic and real groups and also allows to glue together the images of the disjoint regions at real side: this induces gluing at p-adic side. The procedure will be discussed later in more detail.

3. A little technicality is needed. The usual Lie-algebra exponential in the matrix representation contains an imaginary unit. For $p \ mod \ 4 = 3$ this imaginary unit can be introduced as a unit in the algebraic extension. For $p \ mod \ 4 = 1$ it can be realized as an algebraic number. It however seems that imaginary unit or its p-adic analog should belong to an algebraic extension of p-adic numbers. The group parameters for algebraic extension of p-adic numbers belong to the algebraic extension. If the algebraic extension contains non-trivial roots of unity $U_{m,n} = exp(i2\pi m/n)$, the differences $U_{m,n} - U^*_{m,n}$ are proportional to imaginary unit as real numbers and one can replace imaginary unit in the exponential with $U_{m,n} - U^*_{m,n}$. In real context this means only a rescaling of the Lie algebra generator and Planck constant by a factor $(2sin(2\pi m/n))^{-1}$. A natural imaginary unit is defined in terms of U_{1,p^n}.

4. This construction is expected to generalize to the case of coset spaces and give rise to a coset space G/H identified as the union of discrete coset spaces associated with the elements of the coset G_0/H_0 making sense also in the real context. These are obtained by multiplying the element of $G_/H_0$ by the p-adic factor space $G_{p,n}/H_{p,n}$.

One has two hierarchies corresponding to the hierarchy of discrete subgroups of G_0 requiring each some minimal algebraic extension of p-adic numbers and to the hierarchy of G_p:s defined by the powers of p. These two hierarchies can be assigned to angles (actually phases coming as roots of unity) and p-adic length scales in the space of group parameters.

2. Does the hierarchy of Planck constants emerge p-adically?

The Lie algebra of the rotation group spanned by the generators L_x, L_y, L_z provides a good example of the situation and leads to the question whether the hierarchy of Planck constants [K6] could be understood p-adically.

1. Ordinary commutation relations are $[L_x, L_y] = i\hbar L_z$. For the hierarchy of Lie groups it is convenient to extend the algebra by introducing the generators $L_i^{n)} = p^n L_i$ and one obtains $[L_x^{m)}, L_y^{n)}] = i\hbar L_z^{m+n)}$. This resembles the commutation relations of Kac-Moody algebra structurally. Since p-adic integers one the replacement of $\hbar = p^k \rightarrow np^k$, $n \ mod \ p = \neq 0$ produces same Lie-algebra.

2. For the generators of Lie-algebra generated by $L_i^{m)}$ one has $[L_x^{m)}, L_y^{m)}] = ip^m\hbar L_z^{m)}$. One can say that Planck constant is scaled from $\hbar$ to $p^m\hbar$. It is important to realize that $\hbar_{eff} = mp^k\hbar$ for $m \ mod \ p \ \neq 0$ (p-adic unit property) is equivalent with $\hbar_{eff} = p^k\hbar$ in the sense that p-adically the resulting Lie-algebras are same.

3. The earlier proposal assigns the origin of the effective hierarchy of Planck constants $\hbar_{eff} = n\hbar$ to n-furcations of space-time sheets. Recall that n-furcations are assigned with the non-determinism of Kähler action. In n-furcation the solution becomes n-valued meaning the presence of n alternative branches in the usual interpretation. The proposal is that a space-time counterpart of second quantization occurs. Single branch is in the role of single particle state and "classically" the only possible one. "Quantally" also m-branch states, $1 \leq m \leq n$, are allowed. This makes sense in zero energy ontology if the branching occurs either at the space-like ends of the space-time surface inside CD or at light-like wormhole throats. Otherwise one has problem with conservation laws allowing only single branch. The Kähler action for m-branch state would be roughly m times that for single branch states as a sum of the Kähler actions for branches so that one would have $\hbar_{eff} = m\hbar$. This prediction is inconsistent with p-adic Lie-algebra prediction unless $m = p^k$ holds true.

Can these two views about the effective hierarchy of Planck constants be consistent with each other? The connection between p-adic length scale hierarchy and hierarchy of Planck constants has been conjectured already earlier but the recent form of the conjecture is the most quantitive one found hitherto.

1. It a connection exists, it could be due to a relationship between the inherent non-determinism of Kähler action and the generic p-adic non-determinism of differential equations. Skeptic could of course counter-argue that in p-adic context both non-determinisms are present. One can however argue that by the condition that p-adic space-time sheets are maps of real ones and vice versa, these non-determinisms must be equivalent for preferred extremals.

2. Also p-adic non-determinism induces multi-furcations of preferred extremals. These two kinds of multi-furcations should be consistent with each other. Also in p-adic context one can consider "second quantization" allowing simultaneously several branches of multi-furcation. Suppose that the p-adic non-determinism is characterized by integration pseudo-constants (functions with vanishing derivatives), and that the first p^k digits for these functions can be chosen freely. For each integration pseudo-constant involved one would have p^k branches so that for m independent variables there would be p^{mk} branches altogether.

   (a) The argument based on the sum of Kähler actions for n-branch states would suggests $\hbar_{eff} = n\hbar$, $1 \leq n \leq p^{km}$ not consistent with $\hbar_{eff} = p^{mk}\hbar$. Consistency between the two pictures is achieved if all p^{mk} branches are realized simultaneously so that the state is analogous to a full Fermi sphere. This option looks admittedly artificial.

   (b) An alternative possibility is following. Suppose that the p-adic Planck constant is $p^r\hbar$,$r \leq km$, and thus equivalent with $kp^r\hbar$ for all $k \ mod \ p \neq 0$, and that the allowed numbers for branches satisfy $n = n_1p^r \leq p^{mk}$, $n_1 \ mod \ p \neq 0$ so that Planck constant in p-adic sense is equivalent with $p^r\hbar$. This would realize a correspondence between the number of branches of multofurcation and the Planck constant associated with p-adic Lie algebras.

3. Note that also n-adic and even $q = m/n$-adic topology is possible with norms given by powers of integer or rational. Number field is however obtained only for primes. This suggests that if also integer - and perhaps even rational valued scales are allowed for causal diamonds, they correspond to effective n-adic or q-adic topologies and that powers of p are favored.

3. Integration again as the problem

The difficult questions concern again integration. The integrals reduce to sums over the discrete subgroup of G multiplied with an integral over the p-adic variant $G_{p,n}$ of the continuous Lie group. The first integral - that is summation - is number theoretically universal. The latter integral is the problematic one.

1. The easy way to solve the problem is to interpret the hierarchy of continuous p-adic Lie groups $G_{p,n}$ as analogs of gauge groups. But if the wave functions are invariant under $G_{p,n}$, what is the situation with respect to $G_{p,m}$ for $m < n$? Infinitesimally one obtains that the commutator algebras $[G_{p,k}, G_{p,l}] \subset G_p, k+l$ must annihilate the functions for $k + l \geq n$. Does also $G_{p,m}$, $m < n$ annihilate the functions for as a direct calculation demonstrates in the real case. If this is the case also p-adically the hierarchy of groups $G_{p,n}$ would have no physical implications. This would be disappointing.

2. One must however be very cautious here. Lie algebra consists of first order differential operators and in p-adic context the functions annihilated by these operators are pseudo-constants. It could be that the wave functions annihilated by $G_{p,n}$ are pseudo-constants depending on finite number of pinary digits only so that one can imagine of defining an integral as a sum. In the recent case the digits would naturally correspond to powers p^m, $m < n$. The presence of these functions could be purely p-adic phenomenon having no real counterpart and emerge when one leaves the intersections of real and p-adic worlds. This would be just the non-determinism of imagination assigned to p-adic physics in TGD inspired theory of consciousness.

Is there any hope that one could define harmonic analysis in $G_{p,n}$ in a number theoretically universal manner? Could one think of identifying discrete subgroups of $G_{p,n}$ allowing also an interpretation as real groups?

1. Exponentiation implies that in matrix representation the elements of $G_{p,n}$ are of form $g = Id + p^n g_1$: here Id represents real unit matrix. For compact groups like $SU(2)$ or CP_2 the group elements in real context are bounded above by unity so that this kind of sub-groups do not exist as real groups. For non-compact groups like $SL(2,C)$ and T^4 this kind of subgroups make sense also in real context.

2. Zero energy ontology suggests that discrete but infinite sub-groups Γ of $SL(2,C)$ satisfying certain additional conditions define hyperbolic spaces as factor spaces H^3/Γ (H^3 is hyperboloid of M^4 lightcone). These spaces have constant sectional curvature and very many 3-manifolds allow a hyperbolic metric with hyperbolic volume defining a topological invariant. The moduli space of CDs contains the groups Γ defining lattices of H^3 replacing it in finite measurement resolution. One could imagine hierarchies of wave functions restricted to these subgroups or H^3 lattices associated with them. These wave functions would have the same form in both real and p-adic context so that number theoretical universality would make sense and one could perhaps define the inner products in terms of "integrals" reducing to sums.

3. The inclusion hierarchy $G_{p,n} \supset G_{p,n+1}$ would in the case of $SL(2,C)$ have interpretation in terms of finite measurement resolution for four-momentum. If $G_{p,n}$ annihilate the physical states or creates zero norm states, this inclusion hierarchy corresponds to increasing IR cutoff (note that short length scale in p-adic sense corresponds to long scale in real sense!). The hierarchy of groups $G_{p,n}$ makes sense also in the case of translation group T^4 and also now the interpretation in terms of increasing IR cutoff makes sense. This picture would provide a group theoretic realization for with the vision that p-adic length scale hierarchies correspond to hierarchies of length scale measurement resolutions in M^4 degrees of freedom.

4.3.6 What about general coordinate invariance?

In purely algebraic approach one must introduce some preferred coordinate system in which the action of various symmetry transformations is simple: typically induced from linear transformations as in the case of projective spaces. This requires physically preferred coordinate system if one hopes to avoid problems with general coordinate invariance. This approach applies also to more general space-time surfaces. A more general approach would assume general coordinate invariance only modulo finite measurement resolution.

For $H = M^4 \times CP_2$ preferred coordinate systems indeed exist but are determined only apart from the isometries of H. For M^4 the preferred coordinates correspond most naturally to linear Minkowsksi coordinates having simple behavior under isometries. Spherical coordinates are not favored since angles cannot be represented p-adically without infinite-dimensional algebraic extension. For CP_2 complex coordinates in which $U(2) \subset SU(3)$ is represented linearly are preferred. The great virtue of sub-manifold gravity is that preferred space-time coordinates can be chosen as a suitable subset of these coordinates depending on the region of the space-time surface. This reduces the general coordinate transformations to the isometries of the imbedding space but does yet not mean breaking of general coordinate invariance.

Suppose that one accepts the notion of preferred coordinates and assumes that partonic two-surfaces (at least) can be expressed in terms of rational equations (for algebraic extensions rationals are generalized rationals). General coordinate transformations must preserve this state of affairs. GCI must therefore preserve the property of being a ratio of polynomials with rational coefficients. Only those isometries of H are allowed, which respect the algebraic extensions of p-adic numbers used. This means that only a discrete subgroup of isometries can induce general coordinate transformations in p-adic context.

There is however a continuum of choices of preferred coordinates induced by isometries of H so that one obtains a continuum of choices not equivalent under allowed general coordinate transformations. It would seem that general coordinate invariance is broken. The world containing a conscious observer who has chosen coordinate system M_1 differs from the world in which this coordinate system is M_2!

TGD inspired quantum measurement theory leads to this kind of symmetry breaking also in real sector induced by a selection of quantization axis. In TGD framework this choice has a correlate at the level of moduli space of CDs. For instance, the choice of a preferred rest frame forced also by number theoretical vision and construction of preferred extremals would reflect itself in the properties of the interior of the space-time surface even if it need not affect partonic 2-surfaces.

One can argue that it must be possible to realize general coordinate invariance in more general manner than defining physics using preferred coordinates and simple cubic lattice structures for the imbedding space. Maybe also general coordinate invariance should be defined in finite measurement resolution. The lattice structures defining the discretization for imbedding space with non-preferred coordinates would look deformed lattice structures in the preferred coordinates but difference would be vanishing in the pinary resolution used.

5 How to define p-adic manifolds?

What p-adic manifolds are? This is the basic question also in TGD. What p-adic CP_2 could mean, and can one speak about p-adic space-time sheets and about solutions of p-adic field equations in p-adic $M^4 \times CP_2$? Does WCW have p-adic counterpart?

The TGD inspired vision about p-adic space-time sheets as correlates for cognition suggests an approach based on the identification of cognitive representations mapping real preferred extremal to its p-adic counterpart and vice versa in finite pinary resolution so that one would map discrete set of rational points to rational points (rational in algebraic extension of p-adic numbers). One would have real chart leafs for p-adic preferred extremals instead of p-adic ones.

5.1 Algebraic and topological approaches to the notion of manifold

There are two approaches to the notion of manifold and they correspond to the division of mathematics to algebra and topology: some-one has talked about the devil of algebra and angel of topology. In the case of infinite-D WCW geometry and p-adic manifolds the roles of devil and angle seem to however change.

1. In the algebraic approach manifolds are regarded as purely algebraic objects - algebraic varieties - and thus number theoretically universal: only algebraic equations are allowed. Inequalities are not accepted. This notion of manifold is not so general as the topological notion and symmetries play a crucial role. The homogenous spaces associated with pairs of groups and subgroups for which all points are metrically equivalent is a good example about the power of the algebraic

approach made possible by maximal symmetries formulated by Klein as Erlangen program. In the construction of WCW geometry this approach seems to be the only possible one, and gives hopes that infinite-D geometric existence - and thus physics - is unique [K2].

Standard sphere is this approach defined by condition $x^2 + y^2 + z^2 = R^2$ and makes sense in all number fields for rational values of R. Purely algebraic definition is especially suited for defining sub-varieties. Linear spaces and projective spaces are however definable as manifolds purely algebraically. The natural topology for algebraic varieties is so called Zariski topology [A7] in which closed sets correspond to lower-dimensional sub-varieties. TGD can be seen as sub-manifold gravity in $M^4 \times CP_2$ with space-time surfaces identified as preferred extremals characterized purely algebraically: this strongly favors algebraic approach. Algebraic definition of the imbedding space as a manifold and induction of space-time manifold structure from that for imbedding space is also necessary if one wants to define TGD so that it makes sense in all number fields (p-adic space-time sheets are interpreted as correlates for cognition, "thought bubbles").

A correspondence between p-adics and reals is however required and this suggests that purely algebraic approach is not enough.

2. Second - extremely general - approach is topological but works as such nicely only in the real context. Manifolds are constructed by gluing together open n-balls. Here the inequality so dangerous in p-adic context enters the game: open ball consists of points with distance smaller than R from center. Real sphere in this approach is obtained by gluing two disks having overlap around equator.

   In p-adic context this approach fails since p-adic balls are either disjoint or nested. In fact, single point is open ball p-adically so that one can decompose a candidate for a p-adic manifold with p-adic coordinate charts to dust. It turns out that the replacement of p-adic norm with canonical identification resolves the problem and one can induce real topology to p-adic context by using canonical identification to define coordinate charts of the p-adic space-time surface as regions of real space-time surface. The essentially new elements are the use of real coordinate charts instead of p-adic ones and the notion of finite measurement resolution characterized by pinary cutoffs.

5.2 Could canonical identification allow construction of path connected topologies for p-adic manifolds?

The Berkovich approach [A13, A14] is an attempt to overcome the difficulty caused by the weird properties of p-adic balls by adding some points to p-adic balls so that its topology becomes path connected and the original p-adic ball is dense set in the Berkovich ball. Idea is same as in the completion of rationals to reals: new points make rationals a continuum and one can build calculus. I do not understand how Berkovich disks can be glued to manifolds - presumably the path connected topology implies that they can have overlaps without being identical or nested: the overlaps should be through the added points.

The problem of the Berkovich construction is that from physics point of view it looks rather complex: it is difficult to imagine physical realizations for the auxiliary spaces involved with the construction. Also giving up the p-adic topology seems strange since non-Archimedean topology has - to my opinion - a nice interpretation if one considers it as a correlate for cognition.

The Bruhat-Tits tree working for projective spaces does not seem to require completion. Path connectedness is implied by the tree having in well-defined sense projective space as boundary. Points of the p-adic projective space are represented by projective equivalence classes of lattices: this allows to connect the points of p-adic manifold by edge paths and even the notion of geodesic line can be defined.

In the following TGD inspired topological approach to the construction of p-adic manifolds is discussed. The proposal relies on the notion of canonical identification playing central role in TGD and means that one makes maps about p-adic preferred extremal using - not p-adic but *real coordinate charts* defined using canonical identification obeying the crucial triangle inequality. This approach allows also to make p-adic chart maps about real preferred extremals for some values of p-adic prime. The ultrametric norms of Berkovich for formal power series are replaced by Archimedean norms defining coordinate functions and their information content is huge as compared to the Berkovich norms. The hierarchy of length scale resolutions gives rise to a hierarchy of canonical identifications in finite pinary resolution and preferred extremal property allows to complete the discrete image set consisting of rational points to a continuous surface. One can say that path-connectedness at the p-adic side is realized by using discretized paths using induced real topology defined by the canonical identification. This gives a resemblance with Bruhat-Tits tree.

5.2.1 Basic facts about canonical identification

In TGD framework one of the basic physical problems has been the connection between p-adic numbers and reals. Algebraic and topological approaches have been competing also here. The notion of canonical identification solves the conflict between algebra (in particular symmetries) and continuity. Canonical identification combined with the identification of common rationals in finite pinary resolution suggests also a natural replacement of p-adic topology with a path connected effective topology defined as real topology induced to p-adic context by canonical identification used to build real chart leafs.

1. In TGD inspired theory of consciousness canonical identification or some of its variants is a good candidate for defining cognitive representations as representations mapping real preferred extremals to p-adic preferred extremals as also for the realization of intentional action as a quantum jump replacing p-adic preferred extremal representing intention with a real preferred extremal representing action. Could these cognitive representations and their inverses actually define real coordinate charts for the p-adic "mind stuff" and vice versa?

2. In its basic form canonical identification I maps p-adic numbers $\sum x_n p^n$ to reals and is defined by the formula $I(x) = \sum x_n p^{-n}$. I is a continuous map from p-adic numbers to reals. Its inverse is also continuous but two-valued for a finite number of pinary digits since the pinary expansion of real number is not unique ($1 = .999999..$ is example of this in 10-adic case). For a real number with a finite number of pinary digits one can always choose the p-adic representative with a finite number of pinary digits.

3. Canonical identification has several variants. Assume that p-adic integers x are represented as expansion of powers of p^k as $x = p^{rk} \sum x_n p^{kn}$ with $x_0 \neq 0$. One can map p-adic rational number $p^{rk} m/n$ with m and n satisfying the analog of $x_0 \neq 0$ regarded as a p-adic number to a real number using $I_{k,l}^Q$: $I_{k,l}^Q(p^{rk} m/n) \equiv p^{-rk} I_{k,l}(m)/I_{k,l}(n)$.

   In this case canonical identification respects rationality but is ill-defined for p-adic irrationals. This is not a catastrophe if one has finite measurement resolution meaning that only rationals for which $m < p^l, n < p^l$ are mapped to the reals (real rationals actually). One can say that $I_{k,l}^Q$ identifies p-adic and real numbers along common rationals for p-adic numbers with a pinary cutoff defined by k and maps them to rationals for pinary cutoff defined by l. Discrete subset of rational points on p-adic side is mapped to a discrete subset of rational points on real side by this hybrid of canonical identification and identification along common rationals. This form of canonical identification is the one needed in TGD framework.

4. Canonical identification does not commute with rational symmetries unless one uses the map $I_{k,l}^Q(p^{rk} m/n) = p^{-rk} I_{k,l}(m)/I_{k,l}(n)$ and also now only in finite resolution defined by k. For the large p-adic primes associated with elementary particles this is not a practical problem (electron corresponds to $M_{127} = 2^{127} - 1$). The generalization to algebraic extensions makes also sense. Canonical identification breaks general coordinate invariance unless one uses group theoretically preferred coordinates for M^4 and CP_2 and subset of these for the space-time region considered.

5.2.2 The resolution of the conflict between symmetries and continuity

Consider now the resolution of the conflict between algebra and topology in more detail.

1. Algebraic approach suggests the identification of reals and various p-adic numbers along common rationals defined by $I_{\infty,\infty}^Q$ but this correspondence is completely dis-continuous. Therefore one must introduce a finite pinary cutoff p^k so that one maps only integers smaller than p^k to

themselves. Since $I^Q_{k,l}$ does not make sense for p-adic irrationals, one must introduce also second pinary cutoff p^l and use $I^Q_{k,l}$ so that only a finite subset of rational points is mapped to their real counterparts.

2. Topological approach relies on canonical identification and its variants mapping p-adic numbers to reals in a continuous manner. $I_{k,\infty}$ applied to p-adics expressed as $x = p^k u$, $u = \sum x_n p^n$, where u has unit norm, defines such a correspondence. This correspondence does not however commute with the basic symmetries as correspondence along common rationals would do for subgroups of the symmetries represented in terms of rational matrices. Canonical identification fails also to commute with the field equations and the real image fails to be differentiable.

   Finite pinary cutoff ($I^Q_{k,\infty} \to I^Q_{k,l}$) saves the situation. Below the lower pinary cutoff p^k the pseudo-constants of p-adic differential equations would naturally relate to the identification of p-adics and reals along common rationals (plus common algebraics in the case of algebraic extensions).

The notion of finite measurement resolution allows therefore to find a compromise between the symmetries and continuity (that is, algebra and topology). $I^Q_{k,l}$ maps rationals to themselves only up to k pinary digits and the remaining points up to l digits are mapped to rationals but not to themselves. Canonical identification thus maps only a skeleton of manifold formed by discrete point set from real to p-adic context and the preferred extremals on both sides would contain this skeleton. There are many manners to select this rational skeleton, which can also define a decomposition of the real manifold to simplices or more general objects allowing to define homology theory in real context and to induce it to p-adic context so that real homology would be inherited to p-adic context.

5.2.3 Definition of p-adic manifold in terms of canonical identification with pinary cutoff

What is remarkable is that canonical identification can be seen as a continuous generalization of the p-adic norm defined as $N_p(x) \equiv I_{k,l}(x)$ having the highly desired Archimedean property. $I_{k,l}$ is the most natural variant of canonical identification for defining the chart maps from regions p-adic manifold to regions of corresponding real mani-fold (in particular, p-adic preferred extremals to their real counterparts).

1. As already mentioned, one must restrict the p-adic points mapped to real rationals since $I^Q_{k,l}(x)$ is not well-defined for p-adic irrationals having non-unique expression as ratios of p-adic integers. For the restriction to finite rationals the chart image on the real side would consist of rational points. The cutoff means that these rationals are not dense in the set of reals. Preferred extremal property could however allow to identify the chart leaf as a piece of preferred extremal containing the rational points in the measurement resolution used. This would realize the dream of mapping p-adic p-adic preferred extremals to real ones playing a key role in number theoretical universality. When one cannot use preferred extremal property some other constraint would restrict the number of different chart leafs.

2. Canonical identification for the various coordinates defines a chart map mapping regions of p-adic manifold to R^n_+. That each coordinate is mapped to a norm $N_p(x)$ means that the real coordinates are always non-negative. If real spaces R^n_+ would provide only chart maps, it is not necessary to require approximate commutativity with symmetries. Also Berkovich considers norms but for a space of formal power series assigned with the p-adic disk: in this case however the norms have extremely low information content.

3. $I^Q_{k,l}$ indeed defines the analog of Archimedean norm in the sense that one has $N^{k,l}_p(x+y) \leq N^{k,l}_p(x)+N^{k,l}_p(y)$. This follows immediately from the fact that the sum of pinary digits can vanish modulo p. The triangle inequality holds true also for the rational variant of I. $N^{k,l}_p(x)$ is however not multiplicative: only a milder condition $N^{k,l}_p(p^{nk}x) = N^{k,l}_p(p^{nk})N^{k,l}_p((x) = p^{-nk}N^{k,l}_p(x)$ holds true.

4. Archimedean property gives excellent hopes that p-adic space provided with chart maps for the coordinates defined by canonical identification inherits within pinary resolutions real topology and its path connectedness as a discretized version. In purely topological approach forgetting

algebra and symmetries, a hierarchy of induced real topologies would be obtained as induced real topologies and characterized by various norms defined by $I_{k,\infty}$. When symmetries and algebra are brought in, $I_{k,l}^{Q}$ gives a correspondence discretizing the connecting paths. This would give a very close connection with physics.

5. The mapping of p-adic manifolds to real manifolds would make the construction of p-adic manifolds very concrete. For instance, one can map real preferred subset of rational points of a real preferred extremal to a p-adic one by the inverse of canonical identification by mapping the real points with finite number of pinary digits to p-adic points with a finite number of pinary digits. This does not of course guarantee that the p-adic preferred extremal is unique. One could however say that p-adic preferred extremals possesses the topological invariants of corresponding real preferred extremal.

6. The maps between different real charts would be induced by the p-adically analytic maps between the inverse images of these charts. At the real side the maps would be consistent with the p-adic maps only in the discretization below pinary cutoff and could be also smooth.

7. An objection against this approach is the loss of general coordinate invariance. One can however argue that one can require this only within the limits of finite measurement resolution. In TGD framework the symmetries of imbedding space provide a very narrow set of preferred coordinates.

The idea that the discretized version of preferred extremal could lead to preferred extremal by adding new points in iterative manner is not new. I have proposed assuming that preferred extremals can be also regarded as quaternionic surfaces (tangent spaces are in well-defined sense hyper-quaterionic sub-space of complexified octonionic space containing hyper-complex octonions as a preferred sub-space) [K13].

5.2.4 What about p-adic coordinate charts for a real preferred extremal and for p-adic extremal in different p-adic number field?

What is remarkable that one can also build p-adic coordinate charts about real preferred extremal using the inverse of the canonical identification assuming that finite rationals are mapped to finite rationals. There are actually good reasons to expect that coordinate charts make sense in both directions.

Furthermore, if real preferred extremal can be mapped to to p-adic extremals corresponding to two different primes p_1 and p_2, then p_1-adic preferred extremals serves as a chart for p_2-adic preferred extremal and vice versa (one can compose canonical identifications and their inverses to construct the chart maps).

Clearly, real and p-adic extremals define in this manner a category. Preferred extremals are the objects. The arrows are the composites of canonical identification and its inverses mapping to each other preferred extremals belonging to different number fields. This category would be very natural and have profound physical meaning: usually the notion of category tends to be quite too general for the needs of physicist. Category theoretical thinking suggests that full picture of physics is obtained only through this category: this is certainly the case if physics is extended to include physical correlates of cognition and intentionality.

Algebraic continuation from real to p-adic context is one good reason for p-adic chart maps. At the real side one can calculate the values of various integrals like Kähler action. This would favor p-adic regions as map leafs. One can require that Kähler action for Minkowskian and Euclidian regions (or their appropriate exponents) make sense p-adically and define the values of these functions for the p-adic preferred extremals by algebraic continuation. This could be very powerful criterion allowing to assign only very few p-adic primes to a given real space-time surface. This would also allow to define p-adic boundaries as images of real boundaries in finite measurement resolution. p-Adic path connectedness would be induced from real path-connectedness.

In the intersection of real and p-adic worlds the correspondence is certainly unique and means that one interprets the equations defining the p-adic space-time surface as real equations. The number of rational points (with cutoff) for the p-adic preferred extremal becomes a measure for how unique the chart map in the general case can be. For instance, for 2-D surfaces the surfaces $x^n + y^n = z^n$ allow no nontrivial rational solutions for $n > 2$ for finite real integers. This criterion does not distinguish between different p-adic primes and algebraic continuation is needed to make this distinction. The basic condition selecting preferred p-adic primes is that the value of real Kähler/Morse function or its real/imaginary exponent (or both) makes sense also p-adically in some finite-dimensional extension of p-adic numbers.

5.2.5 Some examples about chart maps of p-adic manifolds

The real map leafs must be mutually consistent so that there must be maps relating coordinates used in the overlapping regions of coordinate charts on both real and p-adic side. On p-adic side chart maps between real map leafs are naturally induced by identifying the canonical image points of identified p-adic points on the real side. For discrete chart maps $I_{k,l}^Q$ with finite pinary cutoffs one one must complete the real chart map to - say diffeomorphism. That this completion is not unique reflects the finite measurement resolution.

In TGD framework the situation is dramatically simpler. For sub-manifolds the manifold structure is induced from that of imbedding space and it is enough to construct the manifold structure $M^4 \times CP_2$ in a given measurement resolution (k,l). Due to the isometries of the factors of the imbedding space, the chart maps in both real and p-adic case are known in preferred imbedding space coordinates. As already discussed, this allows to achieve an almost complete general coordinate invariance by using subset of imbedding space coordinates for the space-time surface. The breaking of GCI has interpretation in terms of presence of cognition and selection of quantization axes.

For instance, in the case of Riemann sphere S^2 the holomorphism relating the complex coordinates in which rotations act as Möbius tranformations and rotations around preferred axis act as phase multiplications - the coordinates z and w at Northern and Southern hemispheres are identified as $w = 1/z$ restricted to rational points at both side. For CP_2 one has three poles instead of two but the situation is otherwise essentially the same.

5.3 Could canonical identification make possible definition of integrals in p-adic context?

The notion of p-adic manifold using using real chart maps instead of p-adic ones allows an attractive approach also to p-adic integration and to the problem of defining p-adic version of differential forms and their integrals.

1. If one accepts the simplest form of canonical identification $I(x) : \sum_n x_n p^n \rightarrow \sum x_n p^{-n}$, the image of the p-adic surface is continuous but not differentiable and only integers $n < p$ are mapped to themselves. One can define integrals of real functions along images of the p-adically analytic curves and define the values of their p-adic counterparts as their algebraic continuation when it exists.

   In TGD framework this does not however work. If one wants to define induced quantities - such as metric and Kähler form - on the real side one encounters a problem since the image surface is not smooth and the presence of edges implies that these quantities containing derivatives of imbedding space coordinates possess delta function singularities. These singularities could be even dense in the integration region so that one would have no-where differentiable continuous functions and the real integrals would reduce to a sum which do not make sense.

2. In TGD framework finite measurement resolution realized in terms of pinary cutoffs saves the situation. $I_{k,l}^Q$ is a compromise between the direct identification along common rationals favored by algebra and symmetries but being totally discontinuous without the cutoff l. This cutoff breaks symmetries slightly but guarantees continuity in finite measurement resolution defined by the pinary cutoff l. Symmetry breaking can be made arbitrarily small and has interpretation in terms of finite measurement resolution. Due to the pinary cutoff the chart map applied to various p-adic coordinates takes discrete set of rationals to discrete set of rationals and preferred extremal property can be used to make a completion to a real space-time surface. Uniqueness is achieved only in finite measurement resolution and is indeed just what is needed. Also general coordinate invariance is broken in finite measurement resolution. In TGD framework it is however possible to find preferred coordinates in order to minimize this symmetry breaking.

3. The completion of the discrete image of p-adic preferred extremal under $I_{k,l}^Q$ to a real preferred extremal is very natural. This preferred extremal can be said to be unique apart from a finite measurement resolution represented by the pinary cutoffs k and l. All induced quantities are well defined on both sides.

   p-Adic integrals can be defined as pullbacks of real integrals by algebraic continuation when this is possible. The inverse image of the real integration region in canonical identification defines the p-adic integration region.

4. The integrals of p-adic differential forms can be defined as pullbacks of the real integrals. The integrals of closed forms, which are typically integers, would be the same integers but interpreted as p-adic integers.

It is interesting to study the algebraic continuation of Kähler action from real sector to p-adic sectors.

1. Kähler action for both Euclidian and Minkowskian regions reduces to the algebraic continuation of the integral of Chern-Simons-Kähler form over preferred 3-surfaces. The contributions from Euclidian and Minkowskian regions reduce to integrals of Chern-Simons form over 3-surfaces.

   The contribution from Euclidian regions defines Kähler function of WCW and the contribution from Minkowskian regions giving imaginary exponential of Kähler action has interpretation as Morse function, whose stationary points are expected to select special preferred extremals. One would expect that both functions have a continuous spectrum of values. In the case of Kähler function this is necessary since Kähler function defines the Kähler metric of WCW via its second derivatives in complex coordinates by the well-known formula.

2. The algebraic continuation of the exponent of Kähler function for a given p-adic prime is expected to require the proportionality to p^n so that not all preferred extremals are expected to allow a continuation to a given p-adic number field. This kind of assumption has been indeed made in the case of deformations of CP_2 type extremals in order to derive formula for the gravitational constant in terms of basic parameters of TGD but without real justification [K9].

3. The condition that the action exponential in the Minkowskian regions is a genuine phase factor implies that it reduces to a root of unity (one must have an algebraic extension of p-adic numbers). Therefore the contribution to the imaginary exponent Kähler action from these regions for the p-adicizable preferred extremals should be of form $2\pi(k + m/n)$.

   If all preferred real extremals allow p-adic counterpart, the value spectrum of the Morse function on the real side is discrete and could be forced by the preferred extremal property. If this were the case the stationary phase approximation around extrema of Kähler function on the real side would be replaced by sum with varying phase factors weighted by Kähler function.

   An alternative conclusion is that the algebraic continuation of Kähler action to any p-adic number field is possible only for a subset of preferred extremals with a quantized spectrum of Morse function. One the real side stationary phase approximation would make sense. It however seems that the stationary phases must obey the above discussed quantization rule.

Also holomorphic forms allow algebraic continuation and one can require that also their integrals over cycles do so. An important example is provided by the holomorphic one-forms integrals over cycles of partonic 2-surface defining the Teichmueller parameters characterizing the conformal equivalence class of the partonic 2-surfaces as Riemann surface. The p-adic variants of these parameters exist if they allow an algebraic continuation to a p-adic number. The algebraic continuation from the real side to the p-adic side would be possible on for certain p-adic primes p if any: this would allow to assign p-adic prime or primes to a given real preferred extremal. This justifies the assumptions of p-adic mass calculations concerning the contribution of conformal modular degrees of freedom to mass squared [K3].

5.4 Canonical identification and the definition of p-adic counterparts of Lie groups

For Lie groups for which matrix elements satisfy algebraic equations, algebraic subgroups with rational matrix elements could be regarded as belonging to the intersection of real and p-adic worlds, and algebraic continuation by replacing rationals by reals or p-adics defines the real and p-adic counterparts of these algebraic groups. The challenge is to construct the canonical identification map between these groups: this map would identify the common rationals and possible common algebraic points on both sides and could be seen also a projection induced by finite measurement resolution.

A proposal for a construction of the p-adic variants of Lie groups was discussed in previous section. It was found that the p-adic variant of Lie group decomposes to a union of disjoint sets defined by a discrete subgroup G_0 multiplied by the p-adic counterpart $G_{p,n}$ of the continuous Lie group G. The representability of the discrete group requires an algebraic extension of p-adic numbers. The disturbing feature of the construction is that the p-adic cosets are disjoint. Canonical identification $I_{k,l}$ suggests a natural solution to the problem. The following is a rough sketch leaving a lot of details open.

1. Discrete p-adic subgroup G_0 corresponds as such to its real counterpart represented by matrices in algebraic extension of rationals. $G_{p,n}$ can be coordinatized separately by Lie algebra parameters for each element of G_0 and canonical identification maps each $G_{p,n}$ to a subset of real G. These subsets intersect and the chart-to-chart identification maps between Lie algebra coordinates associated with different elements of G_0 are defined by these intersections. This correspondence induces the correspondence in p-adic context by the inverse of canonical identification.

2. One should map the p-adic exponentials of Lie-group elements of $G_{p,n}$ to their real counterparts by some form of canonical identification.

   (a) Consider first the basic form $I = I_{1,\infty}$ of canonical identification mapping all p-adics to their real counterparts and maps only the p-adic integers $0 \leq k < p$ to themselves.

   The gluing maps between groups $G_{p,n}$ associated with elements g_m and g_n of G_0 would be defined by the condition $g_m I(exp(it_a T^a) = g_n I(exp(iv_a T^a)$. Here t_a and v_a are Lie-algebra coordinates for the groups at g_m and g_n. The delicacies related to the identification of p-adic analog of imaginary unit have been discussed in the previous section. It is important that Lie-algebra coordinates belong to the algebraic extension of p-adic numbers containing also the roots of unity needed to represent g_n. This condition allows to solve v_a in terms of t_a and $v_a = v_a(t_b)$ defines the chart map relating the two coordinate patches on the real side. The inverse of the canonical identification in turn defines the p-adic variant of the chart map in p-adic context. For I this map is not p-adically analytic as one might have guessed.

   (b) The use of $I^Q_{k,l}$ instead of $I = I_{1,\infty}$ gives hopes about analytic chart-to chart maps on both sides. One must however restrict $I^Q_{k,l}$ to a subset of rational points (or generalized points in algebraic extension with generalized rational defined as ratio of generalized integers in the extension). Canonical identification respects group multiplication only if the integers defining the rationals m/n appearing in the matrix elements of group representation are below the cutoff p^k. The points satisfying this condition do not in general form a rational subgroup. The real images of rational points however generate a rational sub-group of the full Lie-group having a manifold completion to the real Lie-group.

   One can define the real chart-to chart maps between the real images of $G_{p,k}$ at different points of G_0 using $I^Q_{k,l}(exp(iv_a T^a) = g_n^{-1} g_m \times I^Q_{k,l}(exp(it_a T^a)$. When real charts intersect, this correspondence should allow solutions v_a, t_b belonging to the algebraic extension and satisfying the cutoff condition. If the rational point at the other side does not correspond to a rational point it might be possible to perform pinary cutoff at the other side.

   Real chart-to-chart maps induce via common rational points discrete p-adic chart-to-chart maps between $G_{p,k}$. This discrete correspondence should allow extension to a unique chart-to-chart map the p-adic side. The idea about algebraic continuation suggests that an analytic form for real chart-to-chart maps using rational functions makes sense also in the p-adic context.

3. p-Adic Lie-groups $G_{p,k}$ for an inclusion hierarchy with size characterized by p^{-k}. For large values of k the canonical image of $G_{p,k}$ for given point of G_0 can therefore intersect its copies only for a small number of neighboring points in G_0, whose size correlates with the size of the

algebraic extension. If the algebraic extension has small dimension or if k becomes large for a given algebraic extension, the number of intersection points can vanish. Therefore it seems that in the situations, where chart-to-chart maps are possible, the power p^k and the dimension of algebraic extension must correlate. Very roughly, the order of magnitude for the minimum distance between elements of G_0 cannot be larger than p^{-k+1}. The interesting outcome is that the dimension of algebraic extension would correlate with the pinary cutoff analogous to the IR cutoff defining measurement resolution for four-momenta.

6 What the notion of path connectedness could mean from quantum point of view?

The notions of open set and path connectedness express something physical but perhaps in a highly idealized form. Canonical identification for preferred extremals provides one promising approach to the challenge of defining path connected topology and at the same time achieving a compromise with symmetries and approximate correspondence via common rationals. The variant I_k^Q for the canonical identification with pinary cutoff can be used to map rational points of the real/p-adic preferred extremal to p-adic/real space-time points to define a skeleton completed to a preferred extremal, which of course need not be unique. In particular, real paths are mapped to p-adic paths in finite pinary cutoff so that the images are always discrete paths consisting of rational points so that the notion of finite pinary resolution is un-avodable.

One could also try to formulate path connectness more microscopically and physically using the tools of quantum physics.

1. The basic point is that there are correlations between different points or physical events associated with different points of manifold. Manifold is more like liquid than dust: one cannot pick up single point from it. In the idealistic description based on real topology one can pick up only open ball. This relates also to finite measurement resolution for lengths: it is not possible to specify single point.

2. Quantum physicist would formulate this in terms of physical correlations. The correlation functions for two fields defined in the manifold are non-vanishing even when the two fields are evaluated at different points.

If one takes the suggestion of quantum physicist seriously, one should reformulate the notion of manifold by bringing in quantum fields and their correlation functions. This approach is alternative to the formulation of p-adic (real) manifold based on real (p-adic) coordinate charts defined by canonical identification.

6.1 Could correlation functions for fermion fields code data about geometric objects?

Quantum TGD suggests another approach to the notion of path connectedness. What could the quantum fields needed to formulate the notion of manifold be in TGD framework? In TGD framework there are only very few choices to consider. Only the induced second quantized fermion fields can be considered in both real and p-adic context. Their correlation functions defined as vacuum expectations of bi-local bilinears are indeed well-defined in both real and p-adic context.

One can define classical bosonic correlation functions for the invariants formed from induce bosonic field but this requires integration over the space-time surface and this might be problematic in p-adic context unless one is able to algebraically continue the real correlation functions to p-adic context. Quantum ergodicity states that these correlation functions characterizing sub-manifold geometry statistically are identical for the space-time surfaces which can appear in the quantum superposition defining WCW spinor field.

1. One could perhaps say:

   Two points are "connected by path!" / have "edge connecting them" as Bruhat and Tits would say / belong to same space-time sheet/partonic 2-surface / belog to two distinct 3-surfaces forming *part of a boundary of the same connected space-time surface ↔ there are non-vanishing fermion-antifermion correlation functions for the point pair in question.*

2. Note that one must consider separately pure right-handed neutrino modes and the remaining spinor modes. For the modified Dirac equation pure right-handed neutrino fields are covariantly constant in CP_2 degrees of freedom and delocalized along entire space-time sheet. In space-time interior the correlation functions for right-handed neutrinos should code for the geometry of the space-time sheet.

   The modes which do not represent pure right-handed neutrinos are restricted to 2-D string world sheets. The conformal correlation functions for the spinor fields restricted to string world sheets should code for the geometry of string world sheets.

3. Everything would reduce to fermionic correlation functions, which in principle are measurable in particle physics experiments. This is in accordance with the general vision of TGD that fermion fields provide all possible information about geometric objects. This would generalize the idea that one can hear the shape of the drum that is deduce the geometry of drum from the correlation functions for sound waves.

4. Real space-time topology would be only a highly idealized description of this physical connectedness, in more physical approach it would be described in terms of fermionic correlation functions allowing to decide whether two points belong to same geometric object or not.

6.2 p-Adic variant of WCW and M-matrix

In zero energy ontology (ZEO) the unitary U-matrix having non-unitary M-matrices are rows and allowing interpretation as "complex" square roots of hermitian density matrices are in key role. The unitary S-matrix appears as a "phase factor" of the "complex" square root and its modulus corresponds to Hermitian square roots of density matrix. What is essential is that M-matrices are multi-local functionals of 3-surfaces defining boundary components of connected space-time surface at the light-like boundaries of causal diamond.

By strong form of holography the information about three-surfaces reduces to data given at partonic 2-surfaces (and their tangent space data). The 3-D boundary components of space-time surface at the boundaries of CD define a coherent unit. The space-time surface takes the role of the path connecting two disjoint 3-surfaces in zero energy ontology and WCW is more like a space formed by multi-points (unions of several disjoint 3-surfaces). Hence the basic difficulty of p-adic manifold theory is circumvented.

Although WCW spinor fields are formally purely classical, the analogs of correlation functions as n-point functions in WCW make sense since the notion of 3-surface is generalized in the manner described above. M-matrix elements serve as building bricks of WCW spinor fields and they are are functionals about the data at partonic 2-surfaces at the boundaries of CD and could have an interpretation as correlation function in WCW giving rise to "path connectedness" in WCW in a number theoretically universal manner.

6.3 A possible analog for the space of Berkovich norms in the approach based on correlation functions

The idea about real preferred extremal as a coordinate chart for p-adic preferred extremal (and vice versa) suggest that canonical identification with cutoff could define naturally p-adic preferred extremal as a path connected space. It would also allow to map preferred real preferred extremals to their p-adic counterparts for some preferred primes and at the same time algebraically continue various quantities such as Kähler action. The hierarchies of pinary cutoffs and resolutions in phase degrees of freedom define a hierarchy of resolutions and the resulting Archimedean norms defined by the the hierarchy of canonical identifications define the analog of the norm space of Berkovich.

Also the idea about correlation functions as counterpart for path connectedness suggests that the ultrametric norm of K-valued field needed to defined Berkovich disk might be replaced with fermionic correlation functions. Could the space of the Berkovich norms have as an analog in this more general approach? The notion of finite measurement resolution seems to lead naturally to this analog also for this option.

One can define the correlation functions in various resolutions. This means varying angle resolution and length scale resolution. Angle resolution -or rather phase resolution in p-adic context - means a hierarchy of algebraic extensions for p-adic number fields bringing in roots of unity $exp(i2\pi/n)$ with increasing values of n. Length scale resolution means increasing number of p-adic primes and CDs with scales given by integer multiples of CP_2 scale.

Fermionic Fock space defines a canonical example about hyper-finite factor of type II_1 (HFF) [K11] and the inclusions of HFFs having interpretation in terms of finite measurement resolution should be involved in the construction. The space of Berkovich norms is replaced with the correlation functions assignable to HFF having fractal structure containing infinite inclusion hierarchies of HFFs.

7 Appendix: Technical aspects of Bruhat-Tits tree and Berkovich disk

In the following more technical aspects of Bruhat-Tits tree and Berkovich disk are discussed.

7.1 Why notions like Bruhat-Tits tree and Berkovich disk?

The constructions like Bruhat-Tits tree and Berkovich disk remain totally incomprehensible unless one understands the underlying motivations. If I have understood correctly, the motivation behind all these strange and complicated looking structures is the attempt to generalize the notion of real manifold to p-adic context using topological approach based on p-adic coordinate maps to p-adic disks which must be completed to Berkovich disks ("disk" could quite well be replaced with "ball").

In the real context manifolds have open balls of R^n defining real topology as building bricks. One glues these balls together along their intersection suitably and obtains global differential structures with various topologies and manifold structures. For instance, sphere can be obtained by gluing two disks having overlap around equator.

In the p-adic context the topology is however totally disconnected meaning that single point is the smallest open set. One cannot build anything coherent from points: they are disjoint or identical unlike the open balls in the real case. More generally: two p-adic balls are either disjoint or either one is contained by another one! No gluing by overlap is possible!

This difficulty has stimulated various theories and Bruhat-Tits tree relates to the theory of Berkovich generalizing the notion of open ball to Berkovich disk [A13, A14] serving as a building brick of p-adic manifolds. The naive p-adic disk is contained as a dense subset to Berkovich disk so that this is like replacing rationals with reals and in this manner gluing them to continuum. Pragmatic physicist is not too enthusiastic about this kind of completions, especially so because the original p-adic topology is replaced with a new one in the completion.

7.2 Technical aspects of Bruhat-Tits tree

The construction of Bruhat-Tits tree for $P^1(Q_p)$ and its generalizations to algebraic extensions can be understood as follows.

1. One must be able to connect any pair of points of $P^1(Q_p)$ by and edge path. The basic building brick of edge path is single edge connecting nearby points of $P^1(Q_p)$. One can start from a simpler situtation first by considering Q_p^2 consisting of points (a,b). If one treats these points just as pairs of p-adic numbers, one cannot do anything. One must represent these pairs as geometric objects in order to define the notion of edge purely set theoretically. The Z_p lattice generated having the pair (a,b) as basis vectors is indeed an object labelled by the pair (a,b). If one wants projective space one must assume that the lattices different by scaling of (a,b) by a non-vanishing p-adic number are equivalent but this is not absolutely essential for the argument.

   Note: Also in TGD one has a space whose points are geometric objects. The geometric object is now 3-surface and the space is the "world of classical worlds" - the space formed by these 3-surfaces.

2. The projective space $P^1(C) = S^2$ has a representation as a coset space $PGl(2,C)/PGl(1,C) \times PGl(2,Z)$. This algebraic relation must generalize by replacing C with Q_p. This means that

$PGl(2, Q_p)$ must act transitively in the set of the geometric objects associated with pairs (a, b). The action on lattices is indeed well-defined and transitive and one can generate all lattices from single lattice defined by the lattice characterized by $(a, b) = (1, 1)$. One has a discrete analog of homogeneous space in the sense that its all points are geometrically equivalent because of the transitive action of $Gl(2, Q_p)$. This reduces the construction to single point, which is an enormous simplification.

Note: Also the construction of the geometry of WCW [K2] in TGD relies on symmetric/homogeneous space property (actually the property of being a union of infinite-dimensional symmetric spaces) making the hopeless task managable by reducing the construction to that at single point of WCW and forcing infinite-dimensional symmetries (symplectic invariance imherited from the boundary of $CD \times CP_2$ and generalization of conformal invariance for light-like 3-surfaces and light-like boundaries of CD). Already in the case of loop spaces [A2] Kähler geometry exists only because of these infinite-dimensional symmetries and is also unique [A10]. One can say that infinite-dimensional Kähler geometric existence is unique.

3. The really important idea is that the internal structure of the point pairs (a, b) allows to define what the existence of "edge" between two nearby points of $P^1(Q_p)$ could mean. The definition is following. Two projective lattices $[M]$ and $[N]$ (projective equivalence classes of lattices) are connected by an edge if there exist representatives M and N such that $M \supset N \subset pM$. Note that this relation holds true only for some representatives, not all. It is also purely set-theoretic.

4. By reducing the situation to the simplest possible case $M \leftrightarrow (a, b) = (1, 1)$ one can easily find the lattices N connected to M. The calculations reduce to the finite field F_p since the inclusion condition implies that $M/pM \supset N/pM \supset pM/pM = \{0\}$ and M/pM is just F_p^2. The allowed N correspond are in one-one correspondence with the F_p subspaces of F_p^2 and there are $p + 1$ of them corresponding to space generated by F_p multiples of $(a, 1)$, $a = 0, ...p - 1$ and $(1, 0)$. Therefore the point $(a, b) = (1, 1)$ is connected to $p+1$ neighbours by single edge. By symmetric space property this is true for all points of $P^1(Q_p)$. The conclusion is that edge paths correspond to a regular tree with valence $p + 1$.

5. $P^1(Q_p)$ is still totally disconnected in p-adic topology. The edge paths however provide $P^1(Q_p)$ with a path-connected topology. The example of Berkovich disk would suggest that one must add to $P^1(Q_p)$ something so that $P^1(Q_p)$ remains a dense subset of this larger structure. The situation would be same as for rationals: rationals become a path connected continuum if one adds all irrational numbers to obtain reals. Rationals define a dense subset of reals and numerics uses only them. In particular, integration becomes possible when irrationals are added. It is however not clear to me whether this kind of completion is needed.

   One can wonder what must be added to the set of Z_p lattices in Q_p^2 or to the set of their projective equivalence classes to build the global differentiable structure. The answer perhaps comes from the observation that the ends of Bruhat-Tits tree correspond to *K-rationals* expressible as ratios of two K-integers - something that numerics can catch at least in real case. Could the completion mean adding also the ends which are K-irrationals? If so then the situation would be very similar to that in TGD inspired definition of p-adic manifolds.

6. Every pair of points in the completion $P^1(Q_p)$ is connected by an edge path consisting of some minimal number n_{min} of edges and this edge path defines the analog of geodesic with length n_{min}. This number is p-adic integer and could be infinite as a real integer for the completion of the p-adic manifold to a path connected manifold. Here the canonical identification $\sum x_n p^n \rightarrow z_n p^{-n}$ mapping p-adic integers to real numbers and playing a key role in p-adic mass calculations could come into play and allow to obtain a real valued finite distance measure. Real distances have continuous spectrum in the interval $[0, p)$. The objection is that this definition is not consistent with the idea of algebraic continuation of integrals from real context.

This construction generalizes to algebraic extensions K of Q_p and also to higher-dimensional projective spaces and symmetric spaces. In particular, the construction of the p-adic counterpart of CP_2 becomes possible. Now one replaces Q_p^2 with Q_p^3 or K^3 allowing the action of some discrete subgroup of the isometry group $SU(3)$ of CP_2. Lattices in K^3 replace the points of Q_p^3 and defines the counterpart of Bruhat-Tits tree in exactly the same manner as for $P^1(K)$.

Physically the highly interesting point is that only a discrete subgroup of CP_2 can be represented in the algebraic extension so that symmetry breaking to discrete subgroup is un-avoidable. In TGD framework the interpretation is in terms of finite measurement resolution forcing discretization and therefore also symmetry breaking. This symmetry breaking is quite different from that defined by Higgs mechanism or symmetry breaking taking place for the solutions of field equations for a variational principle characterized by the unbroken symmetry group.

7.3 The lattice construction of Bruhat-Tits tree does not work for K^n but works for $P^n(K)$: something deep?

The naive expectation is that the construction of Bruhat-Tits tree should work also in the simplest possible case that one can imagine: for p-adic numbers Q_p themselves. The naive guess is that the tree for p-adic numbers with norm bounded by p^n the tree is just the p+1-valent tree with trunk and representing all possible pinary expansions of these p-adic numbers. The lattice construction does not however give this correspondence.

Z_p lattices M in Q_p are parameterized by non-vanishing elements a of Q_p in this case. The multiplication by p-adic integer n of unit norm does not affect a given lattice M a since one has $nka = k_1a$ where n, k, k_1 are p-adic integers. Therefore these lattices are not in one-one correspondence with Q_p but with powers p^n: $|q|_p \leq p^n$ for a given lattice. Therefore the lattice construction fails. It is essential that one considers projective space $P^1(Q)$ instead of Q_p. For Q_p^2 the construction however seems to work.

Note: The condition $M \supset N \supset pM$ for the existence of an edge between two lattices allows only two solutions: the trivial solution $N = M$ and the solution $N = pM$. The counterpart of Bruhat-Tits tree is now 1-valent tree with edges labelled by powers of p.

Also in the case of Q_p^n the correspondence between lattices and points of Q_p^n is 1-to-many since the multiplication by an element of Z_p with unit norm does not affect the lattice. As a matter fact, all elements of Qp^n related by $Sl(n, Q_p)$ correspond to same lattice. Hence the replacement of points with lattices must be restricted to the case of projective spaces.

Physicist might argue that the use of lattices is un-natural and quite too complicated from the point of view of practical physics. I am not sure: it might be that the lattices have some nice physical interpretation and perhaps the outcome - the tree - is more important than the lattices used to achieve it. The fact is that p-adic projective spaces have this kind of "skeleton", and one might well argue that there is no need for the ugly looking completion to a bigger space with path connected and non-ultrametric topology.

In TGD framework the p-adic variants of S^2 and CP_2 are central and the existence of the "skeleton" might be of fundamental significance from the point of view of p-adic TGD and number theoretical universality. Note that S^2 emerges naturally for the light-cone boundary in the case of M^4 ($\delta M^4_+ = S^2 \times R_+$, where R_+ represents light-like radial direction). For M^n, $n \neq 4$, one obtains $S^{k=n-2}$, $k \neq 2$, and this space is not projective space. Also in twistor Grassmannian approach to scattering amplitudes utilizing residue integrals in projective spaces $Gl(n, C)/Gl(n-m, C) \times Gl(m, C)$ this property for the p-adic counterparts of these spaces might be of primary importance.

7.4 Some technicalities about Berkovich disk

Berkovich disk is a p-adic generalization of open ball and meant to serve as a building brick of p-adic manifolds in the same manner as open ball is the building brick of real manifolds. The first guess is that ordinary open ball for p-adic numbers defined by $|x - a| < r$ could work. As a matter fact, p-adic distance is quantized: $|x - a| = p^n$ holds true. The basic outcome of total disconnectedness of the ultrametic topology is that two p-adic balls are either disjoint of the other one is contained by another one. One cannot build manifolds by taking p-adic balls and allowing them to partially overlap to get global differentiable structures and various topologies.

The construction of Berkovich disk - call it B - is motivated by the need to generalize the standard approach to the construction of real manifolds. I do not know whether it is equivalent with the approach based on Bruhat-Tits tree. The explicit realization of Berkovich disk as a completion of ultrametric unit disk is something which one cannot guess easily but when one has understood that the basic premises are satisfied for it, it begins to look less artificial.

I try to explain this construction described briefly in the lecture notes Buildings and Berkovich spaces [A14] by Annette Werner. I neglect all technical issues (I do not even understand them properly!). The basic idea is to imbed ultrametric unit disk as a dense subset to some space possessing path connected topology. The challenge is to guess what this space is.

1. One starts from p-adic unit disk D: $|x|_p \leq 1$, which one wants to complete to Berkovich disk B containing D as a dense subset and possessing path connected topology. One could also replace Q_p with Q_p^n or K^n, where K is any algebraic extension Q_p. In the explanation provided in the lecture notes one considers for simplicity K, which is algebraically complete: this requires an algebraic extension allowing containing all algebraic numbers. This is unrealistic but the construction is possible also for general K but involves more technicalities.

2. One introduces the space of formal K-valued power series $f(z) = \sum f_n z^n$ in $D(0,1) \equiv D$. One can define for the an ultrametric norm as $||f|| = Max\{|f_n|_K\}$. This is actually the supremum of p-adic norm $|f(x)|_K$ in $D(0,1)$. The p-adically largest coefficient f_n defines the norm known as Gauss norm. This norm is multiplicative. For constant functions, which are in one-one correspondence with points of K, this norm reduces to K-norm.

3. One considers also more general norms. In fact, the space of norms with attributes ultrametric, bounded, and multiplicative and reducing for constant functions to K-norm $||_K$ defines the Berkovich unit disk B, which turns out to be a completion of the unit disk D containing D as a dense subset. Furthermore, B turns out to have have path connected topology as required making possible global differentiable structure and even hopes about p-adic integration.

4. Berkovich manages to construct these norms explicitly. The simplest norms of this kind are defined by points a of D. The norm is simply $|f(a)|_K$. These norms are in one-one correspondence with points of D and should define a dense subset of the entire space of norms. The points of K are therefore mapped to subspace of the space of norms: this is absolutely essential.

5. There are also other multiplicative, ultrametric norms reducing to $||_K$ for constant functions in D. They are defined in terms of disks $|x-a|_K \leq r \leq 1$. The Gauss norm corresponds to $r=1$ and the norm described in previous item to $r=0$. These norms are analogous to irrationals numbers in the case of completion of rationals to reals. The Berkovich disk B contains points of four different types.

   - Points of type 1: $|f_a| = |f(a)|_K$ (imbedding of D to Berkovich disk B.
   - Points of type 2: $|f|_{a,r} = sup|f(x)|_K$ for $D(a,r) \subset D(0,1)$ and $r \in |K*|$, the value spectrum of K-norms (powers of p for Q_p). The Gauss norm corresponds to $r=1$.
   - Points of type 3: $|f|_{a,r} = sup|f(x)|_K$ for $D(a,r) \subset D(0,1)$ and $r \notin |K*|$. There is a delicate difference between types 2 and 3 which I fail to understand.
   - Points of type 4: $|f|_{a,r} = lim_{n\rightarrow\infty}|f|_{a_n,r_n}$ for a nested sequence $D(a_1,r_1) \supset D(a_2,r_2)....$ of closed disks in $D(0,1)$.

6. The topology in Berkovich disk is defined by a pointwise convergence of the norm in the space of functions f in D. This topology makes Berkovich disk path connected.

The above construction is rather complicated although and also assumes algebraic completeness. For finite-dimensional algebraic extensions the construction is expected to be even more complicated. I do not understand the possible connection between Bruhat-Tits tree and Berkovich construction: does Bruhat-Tits tree follow from Berkovich construction or not?

7.5 Could the construction of Berkovich disk have a physical meaning?

For the physicist the obvious question is whether the function space associated with the K-disk D could have some some physical interpretation? And what about the interpretation of the space of bounded multiplicative ultrametric norms for this function space? Could these norms have some physical interpretation?

Consider first basic criticism what might be represented by a physicist.

1. The ultrametric multiplicative norms in the function space carry extremely scarce information about the functions. Just the norm of the value of the function at single point. If one wants information in several points on must have a manifold consisting of large minimal number of Berkovich disks. An alternative manner to get information about the function space is to combine the information about all norms.

2. Physicists could also wonder what these K-valued functions are physically. Are they physical fields perhaps? If so, why not consider p-adic variants of correlation functions instead of p-adic norms scalars formed from these fields at single point. This forces however to ask whether the non-vanishing of these physical correlation functions for these fields could code for the existence of "connections" between points of the p-adic manifold so that there would be no need for the completion to Berkovich disk after all. Could the solution of the problem be achieved by bringing quantum physics a part of the definition of the manifold structure.

   It seems that in TGD framework there is no natural counterpart for the K-valued formal power series and their norms. One must perform a stronger generalization and this leads to the use of canonical identification mapping p-adic coordinate variables to their Archimedean norms defined by canonical identification and serving as real coordinates. Another, very speculative approach would be based on correlation functions of fermion fields as a possible manner to code the physical counterpart of path connectedness.

8 Appendix: TGD inspired view about p-adic icosahedron and p-adic quasilattices

The basic criticism against the notion of p-adic icosahedron, and more generally, the notion of p-adic manifold, is the technical complicacy of the constructions. TGD however suggests much simpler construction allowing also a definition of p-adic icosahedron (and Platonic solids) very closely related to its real counterpart.

8.1 TGD based view about p-adic manifolds and icosahedron

The construction of p-adic manifold topology somehow overcoming the difficulty posed by the fact that p-adic balls are either disjoint or nested is necessary. It should also allow a close relationship between p-adic and real preferred extremals. It will be found that TGD leads naturally to a proposal of p-adic manifold topology based on canonical identification used to map the predictions of p-adic mass calculations to real numbers. This map would define coordinate charts for p-adic space-time surfaces - not as p-adic chart leafs as in the standard approach - but as real chart leafs. The real topology induced from real map leafs to the p-adic realm would be path-connected as required.

In TGD framework one must also require finite measurement resolution meaning that the canonical identification is characterized by pinary cutoff takes a discrete subset of rational points of p-adic preferred extremal to its real counterpart: for a subset of this subset rationals are mapped to themselves. One can complete this point set to a real preferred extremal in finite measurement resolution. This construction allows also to define p-adic integrals and differential forms in terms of their real counterparts by algebraic continuation. Therefore geometric notions like distance and volume make sense and there is a very close correspondence between real space-time geometries and their p-adic counterpart in the situations when they exist.

This definition of p-adic manifold topology allows to define projective sphere and Euclidian space E^3 in much simpler manner as in the approach involving Bruhat-Tits tree. Group theoretically this definition is equivalent with the one proposed in the article and algebraic extension of p-adic numbers is required to represent the isometry group of the icosahedron in the p-adic context. What is new is that one can define edges and faces of the Platonic solid as p-adic inverse of the edges and faces on real coordinate chart so that much more concrete representation of icosahedron is obtained. In TGD inspired theory of consciousness p-adic icosahedron would be seen as a cognitive representation of the real icosahedron.

8.2 Can one consider a p-adic generalization of Penrose tiling and quasicrystals?

The mathematically rigorous generalization of Penrose Tilings and quasicrystals to p-adic context might be possible but is bound to be rather technical. The p-adic icosahedron as it is defined in the article does not seem very promising notion. The point is that it is defined in terms of fixed point set for subgroups of icosahedral group acting on Riemann sphere: the action in Euclidian 3-space is now more natural and certainly makes sense and actually simplifies the situation since Q_p^3 sd analog of E^3 is simplest possible 3-D p-adic manifold. It does not however allow Bruhat-Tits tree since the points of Q_p^n are not in 1-1 correspondence with the lattices of Q_p^n. The possibility to construct Bruhat-Tits tree is a special feature of projective spaces.

TGD based view about p-adic E^3 and S^2 as its sub-manifold allows to define also the counterpart of Penrose tiling and QCs in an elegant manner with a close relationship between real and p-adic variants of QC.

1. If one considers lattices in n-dimensional p-adic space Q_p^n replacing E^n, a more natural definition would be in terms of this space than in terms of sphere. For the counterpart of E^3 one can define the action of the subgroup A_5 of rotation group $SO(3)$ by introducing an algebraic extension of the p-adic numbers containing $cos(2\pi/5)$, $sin(2\pi/5)$ and $cos(2\pi/3)$, $sin(2\pi/3)$ and their products. What is interesting is that algebraic extension is forced automatically in p-adic context! In cut and projection method the QC structure requires also this since the imbedded space has an algebraic dimension over integers equal to the dimension of the imbedding space over reals.

   Could it be that p-adic variants of QCs might provide number theoretic insights about QCs? Subspace would define algebraic extension of p-adic numbers and this extension would be such that it allows the representation of the isometry group of the Platonic solid possibly assignable to the QC.

2. One can also now define the icosahedron or any Platonic solid in terms of fixed points also now. Only discrete subgroups of the rotation group can be represented p-adically since algebraic extension is required. This brings in mind the notion of finite measurement resolution leading to a discretization of p-adically representable rotations and more general symmetries. For instance, without algebraic extension only rotations for which the rotation matrices are rational numbers are representable. It seems that finite subgroups of this kind are generated by rotations with rotation angle $\pi/2$ around various coordinate axes. Pythagorean triangles correspond to rationals values of cosine and sine and rotations for which rotation angle corresponds to Pythagorean angle define rational rotation matrices: these groups are discrete but contain infinite number of elements.

   Altogether this suggests a hierarchy of p-adic extensions leading to higher algebraic dimensions and larger discrete symmetries. This conforms with the general number theoretic vision about TGD.

3. Lattices in Q_p^n with integer coefficients make also sense and are characterized by n linearly independent (over p-adic integers) basic vectors $(a_1, ..., a_n)$. Most points of lattice would correspond to values of p-adic integers n_i in $\sum_i n_i a_i$ infinite as real numbers.

   Consider first a non-realistic option in which p-adic integers are mapped to p-adic integers as such. Note also that most of p-adic lattice points would map to real infinity. This kind of correspondence makes sense also for rationals but would give a totally discontinuous correpondence between reals and p-adics.

   p-Adic manifold topology defined in terms of the canonical identification I_{kl} allows to interpret the p-adic lattice as a cognitive representation of the real one. The presence of pinary cutoffs k and l having interpretation in terms of finite cognitive resolution has two implications. Integers $n_i < p^k$ are mapped to themselves so that this portion of lattice is mapped to itself faitfully. The integers $k \leq n < l$ are not mapped to integers and the length of the image is bounded below. The real image of the p-adic lattice under I_{kl} is necessary compressed to a finite volume of E^3. This kind of compression and cutoff is natural for cognitive representations for which numerics with finite cutoff provides one particular analogy.

4. Could the notion of p-adic QC and Penrose tiling make sense if one considers p-adic counterparts of Euclidian space and a n-D cubic lattice with integer valued coefficients and spanned by unit vectors? Could the cut and project method generalize?

   This is not clear since projection would lead from a lattice in Q_p^n to a QC in lower-dimensional space which is associated with algebraic extension of Q_p but having algebraic dimension equal to n. If this space is K^m, K an algebraic extension of Q_p, one has $n = dim(K) \times m$. For prime values of n this would mean that $m = 1$ and one has n-D algebraic extension.

   Projection should be generalized to a map mapping points of n-D space to m-dimensional sub-space K^m associated with algebraic extension of Q_p. Maybe it is better to formally extend Q_p^n to K^n and restrict the lattice to integer lattice in $Q_p^n \subset K^n$. In this manner the projection becomes well-defined as map from $Q_p^n \subset K^n$ to a subspace K^m of K^n. The basic condition could be that the points of the subspace K^m in K^n with algebraic dimension $n \times dim(K)$ define and m-dimensional subspace over K and n-dimensional subspace of Z_p.

   The "irrational angles" associated with the lower-dimensional subspace defining quasilattice defining algebraic extension of Q_p should be such that it allows the representation of the isometry group of the p-adic Platonic solid possibly assignable to the QC in question.

8.3 Cut and project construction of quasicrystals from TGD point of view

Cut and project method is used to construct quasicrystals (QCs) in sub-spaces of a higher-dimensional linear space containing an ordinary space filling lattice, say cubic lattice. For instance, 2-D Penrose tiling is obtained as a projection of part of 5-D cubic lattice - known as Voronyi cell - around 2-D sub-space imbedded in five-dimensional space. The orientation of the 2-D sub-space must be chosen properly to get Penrose tiling. The nice feature of the construction is that it gives the entire 2-D QC. Using local matching rules the construction typically stops.

8.3.1 Sub-manifold gravity and generalization of cut and project method

The representation of space-time surfaces as sub-manifolds of 8-D $H = M^4 \times CP_2$ can be seen as a generalization of cut and project method.

1. The space-time surface is not anymore a linear 4-D sub-space as it would be in cut and project method but becomes curved and can have arbitrary topology. The imbedding space ceases to be linear $M^8 = M^4 \times E^4$ since E^4 is compactified to CP_2. Space-time surface is not a lattice but continuum.

2. The induction procedure geometrizing metric and gauge fields is nothing but projection for H metric and spinor connection at the continuum limit. Killing vectors for CP_2 isometries can be identified as classical gluon fields. The projections of the gamma matrices of H define induced gamma matrices at space-time surface. The spinors of H contain additional components allowing interpretation in terms of electroweak spin and hyper-charge.

8.3.2 Finite measurement resolution and construction of p-adic counterparts of preferred extremals forces "cut and project" via discretization

In finite measurement resolution realized as discretization by finite pinary cutoff one can expect to obtain the analog of cut and project since 8-D imbedding space is replaced with a lattice structure.

1. The p-adic/real manifold structure for space-time is induced from that for H so that the construction of p-adic manifold reduces to that for H.

2. The definition of the manifold structure for H in number theoretically universal manner requires for H discretization in terms of rational points in some finite region of M^4. Pinary cutoffs- two of them - imply that the manifold structures are parametrized by these cutoffs charactering measurement resolution. Second cutoff means that the lattice structure is piece of an infinite lattice. First cutoff means that only part of this piece is a direct imagine of real/p-adic lattice on p-adic/real side obtained by identifying common rationals (now integers) of real and p-adic number fields. The mapping of this kind lattice from real/p-adic side to p-adic/real side defines

the discrete coordinate chart and the completion of this discrete structure to a preferred extremal gives a smooth space-time surface also in p-adic side if it is known on real side (and vice versa).

3. Cubic lattice structures with integer points are of course the simplest ones for the purposes of discretization and the most natural choice for M^4. For CP_2 the lattice is completely analogous to the finite lattices at sphere defined by orbits of discrete subgroups of rotation group and the analogs of Platonic solids emerge. Probably some mathematician has listed the Platonic solids in CP_2.

4. The important point is that this lattice like structure is defined at the level of the 8-D imbedding space rather than in space-time and the lattice structure at space-time level contains those points of the 8-D lattice like structure, which belong to the space-time surface. Finite measurement resolution suggests that all points of lattice, whose distance from space-time surface is below the measurement resolution for distance are projected to the space-time surface. Since space-time surface is curved, the lattice like structure at space-time level obtained by projection is more general than QC.

The lattice like structure results as a manifestation of finite measurement resolution both at real and p-adic sides and can be formally interpreted in terms of a generalization of cut and project but for a curved space-time surface rather than 4-D linear space, and for H rather than 8-D Minkowski space. It is of course far from clear whether one can obtain anything looking like say 3-D or 4-D version of Penrose tiling.

1. The size scale of CP_2 is so small (10^4 Planck lengths) that space-time surfaces with 4-D M^4 projection look like M^4 in an excellent first approximation and using M^4 coordinates the projected lattice looks like cubic lattice in M^4 except that the distances between points are not quite the M^4 distances but scaled by an amount determined by the difference between induced metric and M^4 metric. The effect is however very small if one believes on the general relativistic intuition.

   In TGD framework one however can have so called warped imbeddings of M^4 for which the component of the induced metric in some direction is scaled but curvature tensor and thus gravitational field vanishes. In time direction this scaling would imply anomalous time dilation in absence of gravitational fields. This would however cause only a the compression or expansion of M^4 lattice in some direction.

2. For Euclidian regions of space-time surface having interpretation as lines of generalized Feynman diagrams M^4 projection is 3-dimensional and at elementary particle level the scale associated with M^4 degrees of freedom is roughly the same as CP_2 scale. If CP_2 coordinates are used (very natural) one obtains deformation of a finite lattice-like structure in CP_2 analogous to a deformation of Platonic solid regarded as point set at sphere. Whether this lattice like structure could be seen as a subset of infinite lattice is not clear.

3. One can consider also string like objects $X^2 \times Y^2 \subset M^4 \times CP_2$ with 2-D M^4 projection and their deformations. In this case the projection of M^4 lattice to X^2 - having subset of two M^4 coordinates as coordinates - can differ considerably from a regular lattice since X^2 can be locally tilted with respect to M^4 lattice. This cannot however give rise to Penrose tiling requiring 5-D flat imbedding space. This argument applies also to 2-D string world sheets carrying spinor modes. In the idealized situation that string world sheet is plane in M^4 one might obtain an analog of Penrose tiling but with 4-D imbedding space.

The above quasi lattice like structures (QLs) are defined by a gravitational deformation of the cubic lattice of M^4. Is there any hope about the 4-D QLs in M^4 so that gravitation would give rise to the analogs of phason waves deforming them? Could cut and project method be generalized to give QL in M^4 as projection of 8-D cubic lattice in M^8?

8.3.3 $M^8 - H$ duality

Before considering an explicit proposal I try to describe what I call $M^8 - H$ duality ($H = M^4 \times CP_2$).

1. What I have christened $M^8 - H$ duality is a conjecture stating that TGD can be equivalently defined in M^8 or $M^4 \times CP_2$. This is the number theoretic counterpart of spontaneous compactification of string models but has nothing to do with dynamics: only two equivalent representations of dynamics would be in question.

2. Space-time surfaces (preferred extremals) in M^8 are postulated to be quaternionic sub-manifolds of M^8 possessing a fixed $M^2 \subset M^4 \subset M^8$ as sub-space of tangent space. "Quaternionic" means that the tangent space of M^4 is quaternionic and thus associative. Associativity conditions would thus determine classical dynamics. More generally, these subspaces $M^2 \subset M^8$ can form integrable distribution and they define tangent spaces of a 2-D sub-manifold of M^4. If this duality really holds true, space-time surfaces would define a lattice like structure projected from a cubic M^8 lattice. This of course does not guarantee anything: $M^8 - H$ duality itself suggests that these lattice like structures differ from regular M^4 crystals only by small gravitational effects.

3. The crucial point is that quaternionic sub-spaces are parametrized by CP_2. Quaternionic 4-surfaces of $M^8 = M^4 \times CP_2$ containing the fixed $M^2 \subset M^8$ can be mapped to those of $M^4 \times CP_2$ by defining M^4 coordinates as projections to preferred $M^4 \subset M^8$ and CP_2 coordinates as those specifying the tangent space of 4-surface at given point.

4. A second crucial point is that the preferred subspace $M^4 \subset M^8$ can be chosen in very many manners. This imbedding is a complete analog of the imbedding of lower-D subspace to higher-D one in cut and project method. M^4 can be identified as any 4-D subspace imbedded in M^4 and the group $SO(1,7)$ of 8-D Lorentz transformations defines different imbeddings of M^4 to M^8. The moduli space of different imbeddings of M^4 is the Grassmannian $SO(1,7)/SO(1,3) \times SO(4)$ and has dimension $D = 28 - 6 - 6 = 16$.

   When one fixes two coordinate axes as the real and one imaginary direction (physical interpretation is as an identification of rest system and spin quantization axes), one obtains $SO(1,7)/SO(2) \times SO(4)$ with higher dimension $D = 28 - 1 - 6 = 21$. When one requires also quaternionic structure one obtains the space $SO(1,7)/SU(1) \times SU(2)$ with dimension $D = 28 - 4 = 24$. Amusingly, this happens to be the number of physical degrees of freedom in bosonic string model.

8.3.4 How to obtain quasilattices and quasi-crystals in M^4?

Can one obtain quasi-lattice like structures (QLs) at space-time level in this framework? Consider first the space-time QLs possibly associated with the standard cubic lattice L^4_{st} of M^4 resulting as projections of the cubic lattice structure L^8_{st} of M^8.

1. Suppose that one fixes a cubic crystal lattice in M^8, call it L^8_{st}. Standard M^4 cubic lattice L^4_{st} is obtained as a projection to some M^4 sub-space of M^8 by simply putting 4 Euclidian coordinates for lattice points o constant. These sub-spaces are analogous to 2-D coordinate planes of E^3 in fixed Cartesian coordinates. There are $7!/3!4! = 35$ choices of this kind.

   One can consider also E_8 lattice is an interesting identification for the lattice of M^8 since E_8 is self-dual and defines the root lattice of the exceptional group E_8. E_8 is union of Z^8 and $(Z+1/2)^8$ with the condition that the sum of all coordinates is an even integer. Therefore all lattice coordinates are either integers or half-integers. E_8 is a sub-lattice of 8-D cubic lattice with 8 generating vectors $e_i/2$, with e_i unit vector. Integral octonions are obtained from E_8 by scaling with factor 2. For this option one can imbed L^4_{st} as a sub-lattice to Z^8 or $(Z+1/2)^8$.

2. Although $SO(1,3)$ leaves the imbedded 4-plane M^4 invariant, it transforms the 4-D crystal lattice non-trivially so that all 4-D Lorentz transforms are obtained and define different discretizations of M^4. These are however cubic lattices in the Lorentz transformed M^4 coordinates so that this brings nothing new. The QLs at space-time surface should be obtained as gravitational deformations of cubic lattice in M^4.

3. L^4_{st} indeed defines 4-D lattice at space-time surface apart from small gravitational effects in Minkowskian space-time regions. Elementary particles are identified in TGD a Euclidian space-time regions - deformed CP_2 type vacuum extremals. Also black-hole interiors are replaced with

Euclidian regions: black-hole is like a line of a generalized Feynman diagram, elementary particle in some sense in the size scale of the black-hole. More generally, all physical objects, even in everyday scales, could possess a space-time sheet with Euclidian metric signature characterizing their size (AdS^5/CFT correspondence could inspire this idea). At these Euclidian space-time sheets gravitational fields are strong since even the signature of the induced metric is changed at their light-like boundary. Could it be that in this kind of situation lattice like structures, even QCs, could be formed purely gravitationally? Probably not: an interpretation as lattice vibrations for these deformations would be more natural.

It seems that QLs are needed *already at the level of* M^4. M^8-H duality indeed provides a natural manner to obtain them.

1. The point is that the projections of L^8_{str} to sub-spaces M^4 defined as the $SO(1,7)$ Lorentz transforms of L^4_{st} define generalized QLs parametrized by 16-D moduli space $SO(1,7)/SO(1,3)\times SO(4)$. These QLs include also QCs. Presumably QC is a QL possessing a non-trivial point group just like Penrose tiling has the isometry group of dodecagon as point group and 3-D analog of Penrose tiling has the isometries of icosahedron as point group.

   This would allow to conclude that the discretization at the level of M^8 required by the definition of p-adic variants of preferred extremals as cognitive representations of their real counterparts would make possible 4-D QCs. M^8 formulation of TGD would explain naturally the QL lattices as discretizations forced by finite measurement resolution and cognitive resolution.

   A strong number theoretical constraint on these discretizations come from the condition that the 4-D lattice like structure corresponds to an algebraic extension of rationals. Even more, if this algebraic extension is 8-D (perhaps un-necessarily strong condition), there are extremely strong constraints on the 22-parameters of the imbedding. Note that in p-adic context the algebraic extension dictates the maximal isometry group identified as subgroup of $SO(1,7)$ assignable to the imbedding as the discussion of p-adic icosahedron demonstrates.

2. What about the physical interpretation of these QLs/QCs? As such QLs define only natural discretizations rather than physical lattices. It is of course quite possible to have also physical QLs/QCs such that the points - rather time like edge paths - of the discretization contain real particles. What about a "particle" localized to a point of 4-D lattice? In positive energy ontology there is no obvious answer to the question. In zero energy ontology the lattice point could correspond to a small causal diamond containing a zero energy state. In QFT context one would speak of quantum fluctuation. In p-adic context it would correspond to "though bubble" lasting for a finite time.

3. It is also possible to identify physical particles as edge paths of the 4-D QC, and one can consider time= constant snapshots as candidates for 3-D QCs. It is quite conceivable that the non-trivial point group of QCs favors them as physical QLs.

8.3.5 Expanding hyperbolic tesselations and quasi tesselations obtained by imbedding $H^3 \subset M^4$ to $H^7 \subset M^8$

M^8-$M^4 \times CP_2$ duality and the discretization required by the notion of p-adic manifold relates in an interesting manner to expanding hyperbolic tesselations and quasi tesselations in $H^7 \subset M^8$, and possible expanding quasi-tesselations in obtained by imbedding $H^3 \subset M^4$ to $H^7 \subset M^8$

1. Euclidian lattices E_8, E_7, E_6

I have already considered E_8 lattice in M^8. The background space has however Minkowskian rather than Euclidian metric natural for the carrier space of the E_8 lattice. If one assigns some discrete subgroup of isometries to it, it is naturally subgroup of $SO(8)$ rather than $SO(1,7)$. Both these groups have $SO(7)$ as a subgroup meaning that preferred time direction is chosen as that associated with the real unit and considers a lattice formed from imaginary octonions.

E_8 lattice scaled up by a factor 2 to integer lattice allows octonionic integer multiplication besides sums of points so that the automorphism group of octions: discreted subgroups of $G_2 \subset SO(7)$ would be the natural candidates for point groups crystals or lattice like structures.

If one assumes also fixed spatial direction identified as a preferred imaginary unit, G_2 reduces to $SU(3) \subset SO(6) = SU(4)$ identifiable physically as color group in TGD framework. From this one ends up with the idea about $M^8 - M^4 \times CP_2$ duality. Different imbeddings of $M^4 \subset M^8$ are quaternionic sub-spaces containing fixed M^2 are labelled by points of CP_2.

All this suggests that E_7 lattice in time=constant section of even E^6 lattice is a more natural object lattice to consider. Kind of symmetry breaking scenario $E_8 \rightarrow E_7 \rightarrow E_6 \rightarrow G_2 \rightarrow SU(3)$ is suggestive. This Euclidian lattice would be completely analous to a slicing of 4-D space-time by 3-D lattices labelled by the value of time coordinate and is of course just what physical considerations suggest.

2. Hyperbolic tesselations

Besides crystals defined by a cubic lattice or associated with E_6 or E_7, one obtains an infinite number of hyperbolic tesselations in the case of M^8. These are much more natural in Minkowskian signature and could be also cosmologically very interesting. Quite generally, one can say that hyperbolic space is ideal for space-filling packings defined by hyperbolic manifolds H^n/Γ: they are completely analogous to space-filling packings of E^3 defined by discrete subgroups of translation group producing packings of E^3 by rhombohedra. One only replaces discrete translations with discrete Lorentz transformations. This is what makes these highly interesting from the point of view of quantum gravity.

1. In M^{n+1} one has tesselations of n-dimensional hyperboloid H^n defined by $t^2 - x1^2 - ... - x_n^2 = a^2 > 0$, where a defines Lorentz invariant which for $n = 4$ has interpretation as cosmic time in TGD framework. Any discrete subgroup Γ of the Lorentz group $SO(1,n)$ of M^{n+1} with suitable additional conditions (finite number of generators at least) allows a tesselation of H^n by basic unit H^n/Γ. These tesselations come as 1-parameter families labelled by the cosmic time parameter a. These 3-D tesselations participate cosmic expansion. Of course, also ordinary crystals are crystals only in spatial directions. One can of course discretize the values of a or some function of a in integer multiples of basic unit and assign to each copy of H^n/Γ a "center point" to obtain discretization of M^{n+1} needed for p-adicization.

2. For $n = 3$ one has M^4 and H^3, and this is very relevant in TGD cosmology. The parameter a defines a Lorentz invariant cosmic time for the imbeddings of Robertson-Walker cosmologies to $M^4 \times CP_2$. The tesselations realized as physical lattices would have natural interpretation as expanding 3-D lattice like structures in cosmic scales. What is new is that discrete translations are replaced by discrete Lorentz boosts, which correspond to discrete velocities and observationally to discrete red shifts for distant objecst. Interestingly, it has been found that red shift is quantized along straight lines [**?**]: "God's fingers" is the term used. I proposed for roughly two decades ago an explanation based on closed orbits of photons around cosmic strings [K5]. but explanation in terms of tesselations would also give rise to periodicity. A fascinating possibility is that these tesselation have defined macroscopically quantum coherent structures during the very early cosmology the the size scale of H^3/Γ was very small. One can also ask whether the macroscopic quantum coherence could still be there.

   Hyperbolic manifold property has purely local signatures such as angle surplus: the very fact that there are infinite number of hyperbolic tesselations is in conflict with the the fact that we have Euclidian 3-geometry in every day length scales. In fact, for critical cosmologies, which allow a one-parameter family of imbeddings to $M^4 \times CP_2$ (parameter characterizes the duration of the cosmology) one obtains flat 3-space in cosmological scales. Also overcritical cosmologies for which $a = constant$ section is 3-sphere are possible but only with a finite duration. Many-sheeted space-time picture also leads to the view that astrophysical objects co-move but do not co-expand so that the geometry of time=constant snapshot is Euclidian in a good approximation.

3. Does the notion of hyperbolic quasi-tesselation make sense?

Can one construct something deserving to be called quasi tesselations (QTs)? For QCs translational invariance is broken but in some sense very weakly: given lattice point has still an infinite number of translated copies. In the recent case translations are replaced by Lorentz transformations and discrete Lorentz invariance should be broken in similar weak manner.

If cut and project generalizes, QTs would be obtained using suitably chosen non-standard imbedding $M^4 \subset M^8$. Depending on what one wants to assume, M^4 is now image of M^4_{st} by an element of $SO(1,7)$, $SO(7)$, $SO(6)$ or G_2. The projection - call it P - must take place to M^4 sliced by scaled copies of H^3 from M^8_{st} sliced by scaled copies of H^7/Γ tesselation. The natural option is that P is directly from H^7 to $H^3 \subset H^7$ and is defined by a projecting along geodesic lines orthogonal to H^3. One can choose always the coordinates of M^4 and M^8 in such a manner that the coordinates of points of M^4 are $(t,x,y,z,0,0,0,0)$ with $t^2 - r^2 = a_4^2$ whereas for a general point of H^7 the coordinates are $(t,x,y,z,x_4,...x_7)$ with $t^2 - r^2 - r_4^2 = a_8^2$ for $H^3 \subset H^7$. The projection is in this case simply $(t,x,y,z,x_4,...,x_7) \rightarrow (t,x,y,z,0,...,0)$. The projection is non-empty only if one has $a_4^2 - a_8^2 \geq 0$ and the 3-sphere S^3 with radius $r_4 = \sqrt{a_4^2 - a_8^2}$ is projected to single point. The images of points from different copies of H^7/Γ are identical if S^3 intersects both copies. For r_4 much larger than the size of the projection $P(H^7/\Gamma)$ of single copy overlaps certainly occurs. This brings strongly in mind the overlaps of the dodecagons of Penrose tiling and icosahedrons of 3-D icosahedral QC. The point group of tesselation would be Γ.

4. Does one obtain ordinary H^3 tesselations as limits of quasi tesselations?

Could one construct expanding 3-D hyperbolic tesselations H_3/Γ_3 from expanding 7-D hyperbolic tesselations having H^7/Γ_7 as a basic building brick? This seems indeed to be the outcome at at the limit $r_4 \rightarrow 0$. The only projected points are the points of H^3 itself in this case. The counterpart of the group $\Gamma_7 \subset SO(1,7)$ is the group obtained as the intersection $\Gamma_3 = \Gamma_7 \cap SO(1,3)$: this tells that the allowed discrete symmetries do not lead out from H^3. This seems to mean that the 3-D hyperbolic manifold is H^3/Γ_3, and one obtains a space-filling 3-tesselation in complete analogy for what one obtains by projecting cubic lattice of E^7 to E^3 imbedded in standard manner. Note that $\Gamma_3 = \Gamma_7 \cap SO(1,3)$, where $SO(1,3) \subset SO(1,7)$, depends on imbedding so that one obtains an infinite family of tesselations also from different imbeddings parametrized by the coset space $SO(1,7)/SO(1,3)$. Note that if Γ_3 contains only unit element $H^3 \subset H^7/\Gamma_7$ holds true and tesselation trivializes.

8.3.6 p-Adic variant of the Grassmannian $SO(1,7)/SO(1,3) \times SO(4)$ and Bruhat-Tits tree

p-Adicization requires also to consider the p-adic variants of the Grassmannian $SO(1,7)/SO(1,3) \times SO(4)$. Grassmannians define a generalization of projective spaces and appear in twistor Grassmannian program. According to the article [A8], the construction of the Bruhat-Tits tree generalizes for them. This gives excellent hopes for generalizing the twistor Grassmannian program to p-adic context. Bruhat-Tits tree for $S^2 = SO(3)/SO(2) = P^1(C)$ generalized to $P^1(K)$ (K is any algebraic extension of Q_p) is constructed in terms of projective equivalence classes of integer lattices in K^2 with inclusion relation defining the notion of edge path making possible path connectedness.

In the recent p-adic manifold structure forces 8-D lattices in K^8 and they seem to to take the role of the 2-D lattices K^2. Therefore TGD view about p-adic manifold structure might well be equivalent to the standard view in the case of Grassmannians. For the Grassmannian in question the projective equivalence is replaced with equivalence under $SO(1,3) \times SO(4)$. Therefore one expects that the generalization of Bruhat-Tits tree in 8-D case and its projections to sub-spaces assignable to algebraic extensions K of Q_p appear and correspond to discrete subgroups Γ of $SO(1,7)$. With some additional restrictions on Γ the spaces H^7/Γ define hyperbolic manifolds (H^7 is 7-D hyperboloid in M^8).

This argument makes sense if the counterpart of projective space $P^1(K)$ can be defined also as the analog of $SO(3)/SO(2)$. What looks like a problem is that the "Cartesian" dimension of this space is 2 whereas $P^1(K)$ is 1-D in this sense. The analog of $SO(3)/SO(2)$ can be indeed defined as the Grassmannian $SO(1,2)/Z_p^1 \times SO(2)$ with dimension 1. Z_p^1 denotes the group of p-adic integers with unit norm defining p-adic units analogous to complex phases: they have their inverse as conjugate. What this says that p-adic unit vector 1 is equivalent to any element of Z_p^1. In real context the group of units contains only the real unit so that one obtains Cartesian dimension 2.

8.4 Do Penrose tilings correspond to edge paths of Bruhat-Tits tree for projective sphere $P^1(Q_p)$?

Perhaps it deservers to be mentioned that there is an amusing co-incidence with Penrose tilings (see the book "In search of the Rieman zeros" [A11] by Lapidus, page 200) and between the representation of 2-adic numbers. This representation is in terms of a a tree containing only 3-vertices. Incoming edge represents n:th binary digit in the expansions $x = \sum x_n 2^n$, $x_n = 0, 1$ and the two outgoing edges corresponds to the two values of the $n+1$:th binary digit. Each 2-adic number corresponds to a one particular edge path in this semi-infinite tree. This structure is very much analogous to Bruhat-Tits tree for p-adic projective line.

A given Penrose tiling corresponds to semi-infinite bit string having only non-negative pinary digits and could be seen as a 2-adic integer. Two bit sequences describe same tiling if they differ from each other for a finite number bits only. Could the ends for the analog of Bruhat-Tits tree for p-adic integers (half-infinite paths beginning from some bit) be in one-one correspondence with Penrose tilings! Could one really describe 2-D Penrose tilings 2-adically? What about more general Penrose tilings and QCs? Maybe this conjecture is trivially true since Lapidus, who mentions this description of Penrose tilings, has written his book about p-adic strings [A11].

Unfortunately, I do not understand the arguments leading to the representation of Penrose tilings using bit sequences and whether this co-incidence has some deeper meaning.

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