Some New Ideas Related to Langlands Program *viz.* TGD

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Matti Pitkänen

orcid:0000-0002-8051-4364.
email: matpitka6@gmail.com,
url: http://tgdtheory.com/public_html/,
address: Rinnekatu 2-4 A 8, 03620, Karkkila, Finland.

Abstract

Langlands' program seeks to relate Galois groups in algebraic number theory to automorphic forms and representation theory of algebraic groups over local fields and adeles. Langlands program is described by Edward Frenkel as a kind of grand unified theory of mathematics.

In the TGD framework, $M^8 - M^4 \times CP_2$ duality assigns to a rational polynomial a set of mass shells H^3 in $M^4 \subset M^8$ and by associativity condition a 4-D surface in M^8, and its it to $H = M^4 \times CP_2$. $M^8 - M^4 \times CP_2$ means that number theoretic vision and geometric vision of physics are dual or at least complementary. This vision could extend to a trinity of number theoretic, geometric and topological views since geometric invariants defined by the space-time surfaces as Bohr orbit-like preferred extremals could serve as topological invariants.

Concerning the concretization of the basic ideas of Langlands program in TGD, the basic principle would be quantum classical correspondence (QCC), which is formulated as a correspondence between the quantum states in the "world of classical worlds" (WCW) characterized by analogs of partition functions as modular forms and classical representations realized as space-time surfaces. L-function as a counter part of the partition function would define as its roots space-time surfaces and these in turn would define via Galois group representation partition function. QCC would define a kind of closed loop giving rise to a hierarchy.

If Riemann hypothesis (RH) is true and the roots of L-functions are algebraic numbers, L-functions are in many aspects like rational polynomials and motivate the idea that, besides rationals polynomials, also L-functions could define space-time surfaces as kinds of higher level classical representations of physics.

One concretization of Langlands program would be the extension of the representations of the Galois group to the polynomials P to the representations of reductive groups appearing naturally in the TGD framework. Elementary particle vacuum functionals are defined as modular invariant forms of Teichmüller parameters. Multiple residue integral is proposed as a way to obtain L-functions defining space-time surfaces.

One challenge is to construct Riemann zeta and the associated ξ function and the Hadamard product leads to a proposal for the Taylor coefficients c_k of $\xi(s)$ as a function of $s(s-1)$. One would have $c_k = \sum_{i,j} c_{k,ij} e^{i/k} e^{\sqrt{-1}2\pi j/n}$, $c_{k,ij} \in \{0, \pm 1\}$. $e^{1/k}$ is the hyperbolic analogy for a root of unity and defines a finite-D transcendental extension of p-adic numbers and together with n :th roots of unity powers of $e^{1/k}$ define a discrete tessellation of the hyperbolic space H^2.

This construction leads to the question whether also finite fields could play a fundamental role in the number theoretic vision. Prime polynomial with prime order $n = p$ and integer coefficients smaller than $n = p$ can be regarded as a polynomial in a finite field. If it is irreducible, it defines an infinite prime. The proposal is that all physically allowed polynomials are constructible as functional composites of these.

Contents

1 Introduction

Langlands' program seeks to relate Galois groups in algebraic number theory to automorphic forms and representation theory of algebraic groups over local fields and adeles. Langlands program is described by Edward Frenkel as a kind of grand unified theory of mathematics (https://cutt.ly/1BgbfsL). I have a strong feeling that Langlands program is essential for TGD but every time I encounter the Langlands program, I feel myself an extremely stupid physicist, who tries to understand something, which simply goes over his head. But still I try once again.

1.1 About Langlands program

I am not mathematician enough to really describe Langlands program (https://cutt.ly/ABj2G7D) and its results. I have only a dim idea about the implications of Langlands correspondence and the following is my humble attempt to get some grasp the basic ideas of this immense topic.

1.1.1 Basic ideas

Wikipedia article (https://cutt.ly/ABj2G7D) and the references therein gives a more detailed view of Langlands program [A4, A3], discussed from the TGD perspective in [?, K4]. The following is a brief summary of this article.

1. The slogan "philosophy of cusp forms" was introduced by Harish-Chandra, expressing his idea of a kind of reverse engineering of automorphic form theory, from the point of view of representation theory. Also Israel Gelfand proposed a similar philosophy.

   The discrete completely discontinuous group Γ of $SL(2,R)$ acting in hyperbolic space H^2, fundamental to the classical theory of modular forms, loses its central role. What remains is the basic idea that representations in general are to be constructed by parabolic induction of so-called cuspidal representations.

   Cuspidal representations assignable to hyperbolic 2-manifolds and their higher-D generalizations, of which Teichmueller spaces as moduli spaces of conformal equivalence classes of Riemann surfaces represent an example, become the fundamental class of objects, from which other representations may be constructed by procedures of induction. Note that in TGD, hyperbolic 3-manifolds could replace hyperbolic 2-manifolds and one challenge is to understand how hyperbolic 2-manifolds relate to hyperbolic 3-manifolds.

   Remark: Cusps correspond geometrically to peak-like singularities of say $SL(2,R)/\Gamma$. Parabolic group (https://cutt.ly/HBj4t4e) is a subgroup of a linear algebraic group G in field k such that G/P is a projective algebraic variety and contains some Borel subgroup of G as a subgroup (upper diagonal matrices with units at diagonal is the standard example).

2. Functoriality as a category theoretic notion is the second key notion. Roughly, functoriality means that what holds true for a representative of a given type group, should hold generally. This makes the statements extremely general. The statements can be formulated in adelic framework so that they hold simultaneously for both rationals, extensions of rationals and extensions of p-adic number fields induced by them.

1.1.2 Contents of Langlands conjectures

1. Langlands correspondence is between L-functions associated with irreps of finite Galois group analogous to zeta functions and automorphic cuspidal representations of $Gl(n,C)$ and of even more general reductive groups representable as matrix groups which are analogous to partition functions. Both partition functions and L-functions code for the numbers of objects of particular kind, typically for the degeneracies of quantum states with given quantum numbers.

$SL(2,C)$ as a covering of Lorentz group is of special interest in TGD but TGD involves many other reductive groups and partition function type objects could define analogies of automorphic forms, which Langlands correspondence maps to L-functions, which are conjectured to satisfy Riemann hypothesis and functional equations analogous to that satisfied by Riemann ζ.

2. In the case of Artin function L-function is a characteristic determinant for an special element of Galois group, which is Frobenius element mapping elements of the ring of integers of L/K to their p:th power: $x \rightarrow x^p$. For finite fields, $x^p = x$ holds true.

   The Artin conjecture states that automorphic forms (`https://cutt.ly/qBgb6Fw`) as representations of reductive groups correspond to Artin L-functions (`https://cutt.ly/NBgnozT`) assigned to Galois groups and having a product representation analogous to the Euler product for ζ. Artin zeta function is a product of powers of Artin L-functions for all finite-D irreducible representations of the Galois group (see Appendix).

   Langlands pointed out that the Artin conjecture follows from strong enough results implied by the Langlands philosophy, relating to the L-functions associated to automorphic representations for $GL(n)$ for all $n \geq 1$.

3. More precisely, the Langlands correspondence associates an automorphic representation of the adelic version of an algebraic group $GL_n(A_Q)$ to every n-dimensional irreducible representation of the Galois group. The automorphic representation is a cuspidal representation (the representation functions vanish at the tips of cusps) if the Galois representation is irreducible. The Artin L-function of the Galois representation is the same as the automorphic L-function of the automorphic representation. Therefore finite-D representations of Galois group and cuspidal representations of $Gl(n, A_Q)$ correspond to each other.

   The Artin conjecture follows immediately from the known fact that the L-functions of cuspidal automorphic representations are holomorphic. This was one of the major motivations for Langlands' work.

4. Dedekind conjecture states that if L/K is an extension of number fields, then the quotient $s \mapsto \zeta_L(s)/\zeta_K(s)$ of their Dedekind zeta functions is entire function. The Aramata-Brauer theorem states that the conjecture holds if L/K is Galois.

5. There are a number of related Langlands conjectures. There are many different groups over many different fields for which they can be stated, and for each field there are several different versions of the conjectures.

There are different types of objects for which the Langlands conjectures can be formulated.

1. Representations of reductive groups over local fields, that is archimedean local fields, p-adic local fields, and completions of function fields over complex numbers). In the case of algebraic groups over local fields, adeles allow to combine the representations in all these fields to a single adelic representation, which implies huge generality.

2. Automorphic forms on reductive groups over global fields, which are extensions of rationals or to a function field over finite field defined by rational functions.

3. Representations of reductive groups over finite fields.

1.2 Why Langlands program could be relevant for TGD?

It is increasingly clear that the conjectures of the Langlands program have physical analogies in the quantum TGD proposed to be a grand unification of physics.

1. In the view of TGD based on fusion number theoretical and geometric views of physics, rational polynomials determine space-time regions at the fundamental level [L5, L6]. The observations of [L15, L12] inspired the question whether L-functions as generalizations of polynomials be used to define space-time surfaces.

Conformal confinement would favor this [L12]. The hypothesis that roots are algebraic numbers becomes an interesting possibility strongly favored by Galois confinement implying that the 4-momenta of physical states have integer components whereas virtual states have momenta with algebraic integer valued components. Momentum components would be algebraic integers in an infinite-D extension of rationals.

What could be the interpretation of these surfaces? Could they represent a higher level of intelligence and define infinite cognitive representations as algebraic integer valued virtual momenta at the mass shells of $M^4 \subset M^8$?

2. Artin's L-functions are associated with n-D representations of Galois groups on one hand and with infinite-D unitary representations ($Gl(n,C)$ and more general Lie groups. The extensions of the representations of Galois groups would be very relevant in TGD since Galois groups become symmetry groups in the number theoretic vision of TGD.

   Quantum TGD provides several candidates for these kinds of groups [L15]. There are groups assignable to the representations of supersymplectic algebras, isometry algebras of the light-cone boundary δM^4_+, and the Kac-Moody type algebras assignable to light-like 3-surfaces defining either boundaries of Minkowskian regions or orbits of partonic 2-surfaces as boundaries between Minkowskian and Euclidean space-time regions [L14]. There are also extended conformal symmetries due to the fact that the light-cone boundary and light-like 3-surfaces are metrically 2-D.

3. The mass shells H^3 of causal diamond (CD) defined by the roots of polynomials allow a realization of $SO(1,3)$ and $SL(2,C)$ allow tessellations and hyperbolic manifolds as analogs of unit cells of lattice. They could make possible the realization of holographic continuations of modular forms associated with hyperbolic 2-manifolds defining boundaries of 3-D hyperbolic manifolds, which could be mapped to L-functions, possibly defining space-time surfaces as analogies of polynomials [L9].

4. Elementary particle vacuum functionals are analogous to partition functions and are determined as modular invariant modular forms in the Teichmueller space parameterizing the conformal equivalence classes of partonic 2-surfaces [K2]. These functions should define L-functions with several variables and they could give rise to L-functions of a single variable by multiple residue integral. For multiple-zetas this procedure gives a product expressible in terms of zetas having the desired physical properties (allowing conformal confinement and possibly even Galois confinement).

1.3 Quantum classical correspondence as a feedback loop between the classical space-time level and the quantal WCW level?

Quantum classical correspondence (QCC) has been one of the guidelines in the development of TGD but its precise formulation has been missing. A more precise view of QCC could be that there exists a feedback loop between classical space-time level and quantal "world of classical worlds" (WCW) level. This idea is new and akin to Jack Sarfatti's idea about feedback loop, which he assigned with the conscious experience. The difference between consciousness and cognition at the human *resp.* elementary particle level could correspond to the difference between L-functions and polynomials.

This vision inspires the question whether the generalization of the number theoretic view of TGD so that besides rational polynomials (subject to some restrictions) also L-functions, which have a nice physical interpretation if RH holds true for them, can be defined via their roots 4-surfaces in M^8_c and by M^8-H duality 4-surfaces in H. Both conformal confinement (in weak and strong form) and Galois confinement (having also weak and strong form) support the view that L-functions are Langlands duals of the partition functions defining quantum states.

If L functions indeed appear as a generalization of polynomials and define space-time surfaces, there must be a very deep reason for this.

1. The key idea of computationalism is that computers can emulate/mimic each other. Universe should be able to emulate itself. Could WCW level and space-time level mimic each other?

If this were the case, it could take place via QCC. If so, it should be possible to assign to a quantum state a space-time surface as its classical space-time correlate and vice versa.

2. There are several space-time surfaces with a given Galois group but fixing the polynomial P fixes the space-time surface. An interesting possibility is that the observed classical space-time corresponds to superposition of space-time surfaces with the same discretization defined by the extension defined by the polynomial P. If so, the superposition of space-time surfaces would be effectively absent in the measurement resolution used and the quantum world would look classical.

3. A given polynomial P fixes the mass shells $H^3 \subset M^4 \subset M^8$ but does not fix the space-time surface X^4 completely since the polynomial hypothesis says nothing about the intersections of X^4 with H^3 defining 3-surfaces. The associativity hypothesis for the normal space of $X^4 \subset M^8$ [L5, L6] implies holography, which fixes X^4 to a high degree for a given X^3. Holography is not expected to be completely deterministic: this non-determinism is proposed to serve as a correlate for intentionality.

   If space-time has boundaries, the boundaries X^2 of $X^3 \subset H^3$ could be ends of light-like 3-surfaces X^3_L [L14]. An attractive idea is that they are hyperbolic manifolds or pieces of a tessellation defined by a hyperbolic manifold as the analog of a unit cell [L9]. The ends X^2 of these 3-surfaces at the boundaries of CD would define partonic 2-surfaces.

   By quantum criticality of the light-like 3-surfaces satisfying $det(-g_4) = 0$ [L14], their time evolution is not expected to be completely unique. If the extended conformal invariance of 3-D light-like surfaces is broken to a subgroup with conformal weights, which are multiples of integer n the conformal algebra defines a non-compact group serving as a reductive group allowing extensions of irreps of Galois group to its representations.

   One can also consider space-time surfaces without boundaries. They would define coverings of M^4 and there would be several overlapping projections to H^3, which would meet along 2-D surfaces as analogies of boundaries of 3-space. Also in this case, the idea that the X^3 is a hyperbolic 3-manifold is attractive.

4. Quantum TGD involves a general mechanism reducing the infinite-D symmetry groups to finite-D groups, which has an interpretation in terms of finite measurement resolution [L15] describable both in terms of inclusions of hyperfinite factors of type II_1 and inclusions of extensions of rationals inducing inclusions of cognitive representations. One can also consider an interpretation in terms of symmetry breaking.

   This reduction means that the conformal weights of the generators of the Lie-algebras of these groups have a cutoff so that radial conformal weight associated with the light-like coordinate of δM^4_+ is below a maximal value n_{max}. The generators with conformal weight $n > n_{max}$ and their commutators with the entire algebra would act like a gauge algebra, whereas for $n \leq n_{max}$ they generate genuine symmetries. The alternative interpretation is that the gauge symmetry breaks from $n_{max} = 0$ to $n_{max} > 0$ by transforming to dynamical symmetry.

   Note that the gauge conditions for the Virasoro algebra and Kac-Moody algebra are assumed to have $n_{max} = 0$ so that a breaking of conformal invariance would be in question for $n_{max} > 0$.

5. The natural expectation is that the representation of the Galois group for these space-time surfaces defines representations in various degrees of freedom in terms of the semi-direct products of the Langlands duals ${}^LG^0$ with the Galois group (here ${}^LG^0$ denotes the connected component of Langlands dual of G). Semi-direct product means that the Galois group acts on the algebraic group G assignable to algebraic extension by affecting the matrix elements of the group element.

   There are several candidates for the group G [L15]. G could correspond to a conformal cutoff A_n of algebra A, which could be the super symplectic algebra SSA of $\delta M^4 \times CP_2$, the infinite-D algebra I of isometries of δM^4_+, or the algebra $Conf$ extended conformal symmetries of $\delta M4_+$. Also the extended conformal algebra and extended Kac-Moody type algebras of H isometries associated with the light-like partonic orbits can be considered.

6. One could assign to these representations modular forms interpreted as generalized partition functions, kind of complex square roots of thermodynamic partition functions. Quantum TGD can be indeed formally regarded as a complex square root of thermodynamics. This partition function could define a ground state for a space of zero energy state defined in WCW as a superposition over different light-like 3-surfaces.

These considerations boil down to the following questions.

1. Could the quantum states at WCW level have classical space-time correlates as space-time surfaces, which would be defined by the L-functions associated with the modular forms assignable to finite-D representations of Galois group having a physical interpretation as partition functions?

2. Could this give rise to a kind of feedback loop representing increasingly higher abstractions as space-time surfaces. This sequence could continue endlessly. This picture brings in mind the hierarchy of infinite primes [L15].

   Many-sheeted space-time would represent a hierarchy of abstractions. The longer the scale of the space-time sheet the higher the level in the hierarchy.

1.4 TGD analogy of Langlands correspondence

Concerning the concretization of the basic ideas of Langlands program in TGD, the basic principle would be quantum classical correspondence (QCC).

1. QCC is formulated as a correspondence between the quantum states in WCW characterized by analogs of partition functions as modular forms and classical representations realized as space-time surfaces. L-function as a counter part of the partition function would define as its roots space-time surfaces and these in turn would define via finite-dimensional representations of Galois groups partition functions. Finite-dimensionality in the case of L-functions would have an interpretation as a finite cognitive and measurement resolution. QCC would define a kind of closed loop giving rise to a hierarchy.

2. If Riemann hypothesis (RH) is true and the roots of L-functions are algebraic numbers, L-functions are in many aspects like rational polynomials and motivate the idea that, besides rationals polynomials, also L-functions could define space-time surfaces as kinds of higher level classical representations of physics.

3. One should construct Riemann zeta and the associated ξ function as the simplest instances of L-functions assignable to $SL(2,R)$. The Hadamard product leads to a proposal for the Taylor coefficients c_k of $\xi(s)$ as a function of $s(s-1)$. One would have $c_k = \sum_{i,j} c_{k,ij} e^{i/k} e^{\sqrt{-1}2\pi j/n}$, $c_{k,ij} \in \{0, \pm 1\}$. $e^{1/k}$ is the hyperbolic analogy for a root of unity and defines a finite-D transcendental extension of p-adic numbers and together with n :th roots of unity powers of $e^{1/k}$ define a discrete tessellation of the hyperbolic space H^2 (upper complex plane). Thus the proposal that mass squared values correspond algebraic numbers generalizes: also roots of e can appear as roots.

4. One concretization of Langlands program would be the extension of the representations of the Galois group to the polynomials P to the representations of reductive groups appearing naturally in the TGD framework [L15].

5. In particular, elementary particle vacuum functionals are defined as modular invariant forms of Teichmüller parameters [K2]. Multiple residue integral is proposed as a way to obtain L-functions defining space-time surfaces.

6. A highly interesting feedback to the number theoretic vision emerges. The rational polynomials P defining space-time surfaces are characterized by ramified primes. Without further conditions, they do not correlate at all with the degree n of P as the physical intuition suggests.

In [L15] it was proposed that P can be identified as the polynomial Q defining an infinite prime [K6]: this implies that P is irreducible.

An additional condition is that the coefficients of P are smaller than the degree n of P. For $n = p$, P could as such be regarded as a polynomial in a finite field. This proposal is too strong to be true generally but could hold true for so-called prime polynomials of prime order having no functional decomposition to polynomials of lower degree [A2, A8]. The proposal is that all physically allowed polynomials are constructible as functional composites of these. Also finite fields would become fundamental in the TGD framework.

2 Langlands conjectures in the TGD framework?

$M^8 - H$ duality is a central element of TGD and states the duality of number theoretic and geometric views of physics. This duality is very analogous to Langlands duality.

2.1 How Langlands duality could be realized in TGD

It has become gradually more and more clear that the conjectures of the Langlands program could be an essential part of quantum TGD [L8, L15, L14, L12, L9] proposed as a candidate for a grand unification of physics.

1. Could L-functions as generalizations of polynomials be used to define space-time surfaces? The generalization of Riemann hypothesis (RH) states that the non-trivial zeros of L-functions are at critical line and trivial ones at negative real axis. This makes possible conformal confinement in both weak form (conformal weight is integer) and strong form (the sum positive and negative (tachyonic) conformal weights vanishes [L16]). The hypothesis that the roots of L-function are algebraic numbers in an infinite-D extension of rationals is the simplest conjecture and allows the realization of Galois confinement so that the 4-momenta have integer valued momenta using the unit defined by the scale of CD. The transcendental extensions by roots of e define finite-D extensions of p-adic numbers and could also be involved.

   What could be the interpretation of these surfaces? Could they represent higher level of intelligence, could they define infinite cognitive representations.

2. Artin's L-functions are associated with n-D representations of Galois groups on one hand and with infinite-D unitary representations ($Gl(n,C)$ and more general Lie groups. $GL(n,C)$ generalizes to n-dimensional reductive group of which $SL(n), SO(k,n-k)$, and $Sp(2n)$ are examples

   The general proposal [L15] is that the super-symplectic algebra assignable to $\delta M^4_+ \times CP_2$ defining the boundary of causal diamond (CD) in zero energy ontology (ZEO) acts as isometries of WCW.

   The dimension 3 of the light-cone boundary makes possible conformal transformations of $S^2 \subset \delta M^4_+$ made local with respect to the light-like radial coordinate of δM^4_+ and CP_2 as candidates for symmetries. As a special case, one has isometries of $\delta M^4_+ \times CP_2$ for which the local conformal scaling from conformal transformation of S^2 is compensated by a scaling for the radial light-like coordinate depending also on $S^2 \times CP_2$ coordinates are possible symmetries.

   The light-like partonic orbits as boundaries between Minkowskian and Euclidean regions and more general light-like boundaries of space-time surfaces are metrically 2-D and allow generalization of conformal symmetries and possibly also Kac-Moody symmetries assignable to isometries as candidates for symmetries.

   All these algebras, denote them by A, allow infinite-D Lie-algebra labelled by radial conformal weights containing as sub-algebras a hierarchy of sub-algebras A_n for which the conformal weights come as n-multiples of the conformal weights of the entire algebra.

   The states spaces annihilated algebra A_n and the commutator $[A_n, A]$ define a hierarchy of state spaces generalizing the state space for which entire algebra annihilates the states. The associated groups would allow a realization of Langlands groups.

3. $n=2$-D representations could be assigned with complex 2-D representations $SL(2,C)$ at the mass shells H^3 defined by the roots of L-function. The tessellations defined by the discrete completely discontinuous subgroups of $SL(2,C)$ would give rise to hyperbolic manifolds as analogs of unit cells for lattices [L9]. One can also associated with them modular forms which would be mapped to L-functions. Fermionic spin could provide these representations.

   The represention of Galois group sould be somehow extended to a representation of $SL(2,C)$. Could this give a connection between the number theoretical physics of TGD involving the irreps of Galois groups and spinor representations of Lorentz group at mass shells H^3?

4. Elementary particular vacuum functionals as analogs of partition functions in modular degrees of freedom of partonic 2-surface are central for TGD view of family replication phenomenon and can be regarded as modular invariants. They could be mapped to the analogs of L-functions of m arguments. Symmetrized multiple-zetas decompose to a sum over products of ordinary zetas. $m-1$-fold residue integral would give something proportional to ordinary ζ satisfying RH and could define space-time surface as a correlate of the corresponding quantum states.

2.2 Could quantum classical correspondence define an infinite hierarchy of abstractions?

The realization of QCC between WCW and classical levels, proposed in the introduction, gives rise to a hierarchy of space-time sheets with increasing algebraic complexity possibly related also to the hierarchy of infinite primes. Schematically one has the following hierarchy.

Polynomial P *rightarrow* space-time surface with Galois group $\rightarrow$ partition function $Z \rightarrow$ L-function $\rightarrow$ space-time surface with Galois group $Gal_L \rightarrow$ ... There are however strong bounds to the complexity which is representable at quantum level.

It is not easy to imagine the complexity at the higher levels of the hierarchy.

1. If one can speak of a Galois group Gal_L of L-function, it is infinite but profinite and has an ultrametric topology, presumably consisting of p-adic sectors p (there is an analogy with the energy landscape of spin glasses in the TGD view of them [L7]).

   It is not enough that the space-time surface defined by L-function contains information of quantum state, this information must be also represented as quantum state and this requires a new partition function assigned with Gal_L. This suggests a connection with the hierarchy of infinite primes [K6] analogous to a hierarchy of second quantizations of a supersymmetric arithmetic QFT [L15]. The assumption that the representations of Gal_L are finite-dimensional would pose a strong constraint to the complexity.

2. For composite polynomials $P_n \circ .... \circ P_1$, the Galois group Gal has a decomposition to a hierarchy of normal subgroups such that a normal subgroup H is Galois group for an extension rationals. The group representation reduced to that for H if Gal/H is represented trivially. If also Gal_L has finite normal subgroup H, one obtains finite-D representations by requiring that Gal/H is represented trivially. This would mean a huge loss of information.

   Can Gal_L have finite normal subgroups? If the L-function is determined by a partition function associated with a representation of Gal, Gal itself is a good guess for H so that Gal_L would reduce to Gal in this particular case! This would conform with the idea that the higher levels of the hierarchy contain all the lower levels.

What one can say about the Galois group Gal_L having variants for rationals and various p-adic number fields.

1. The Absolute Galois group (`https://cutt.ly/nBgndkY`) assignable to algebraic numbers acts as automorphisms of algebraic numbers leaving rationals invariant. This definition could apply also in the case of L-function even in the case that the extension of rationals assigned to L-function involves transcendentals.

   For rationals Absolute Galois group is infinite but profinite, which says that it is in some sense composed of finite groups. Profinite topology is totally discontinuous as also p-adic topology (hthttps://cutt.ly/MBxdFg8). A system of finite groups and homomorphisms between them is needed and implies that finite approximations are excellent. Profiniteness is analogous to hyperfiniteness for the factors of von Neumann algebras, which are central in quantum TGD [L15], and are indeed assumed to be closely related to the hierarchies of extensions of rationals.

2. The absolute Galois groups for finite-D extensions K of p-adic number field Q_p have a finite number of elements given by $N = K/Q_p + 3$ so that in p-adic sectors the situation simplifies dramatically, and this reduction would naturally be behind the profiniteness. This must be essential also in the case of the absolute Galois group of rationals and its extensions.

3. Galois groups of infinite-D extensions, say those possibly associated with L-functions, are also profinite.

Suppose that one can speak of the Galois group Gal_L of an L-function associated with a finite Galois group Gal. Suppose Gal_L has finite subgroups, such as Gal.

1. Could this kind of finite-D representation for Gal_L be assigned with, not a necessary rational polynomial, of finite degree? Galois group can indeed permute also the roots of a polynomial, which is not rational. Now one does not however obtain a finite-D extension of rationals.

2. For instance, the cutoff of the product representation of ξ function (https://cutt.ly/5BjcCcv) associated with ζ as a product $\xi(s) = \prod_k (1 - s/s_k)(1 - s/\overline{s}_k)$, assuming that the imaginary parts of the roots are below some upper bound, defines a polynomial P, which is not a rational polynomial and has coefficients, which belong to an extension of rationals, which need not be finite-D or even algebraic. The roots of the polynomial define an extension of this extension. It is implausible that the extension defined by a finite number of roots of ζ can be a finite-D extension of rationals.

   This leads to an interesting, possibly testable, conjecture concerning $\xi(s) \equiv \tilde{\xi}(u = s(s-1)) = \sum_k c_k u^k$ and its generalization for the extensions of rationals. Complete p-adic democracy requires that the coefficients have the same meaning irrespective of the number field. This is true if the Taylor coefficients c_k of $\tilde{\xi}(u)$ satisfy $c_k = \sum_{i,j} c_{k,ij} e^{i/k} e^{\sqrt{-1}2\pi j/n}$, $c_{k,ij} \in \{0, \pm 1\}$. $e^{1/k}$ defines the hyperbolic analogy for a root of unity and gives rise to a *finite-D* transcendental extension of p-adic numbers. Together with n :th roots of unity powers of $e^{1/k}$ define a discrete tessellation of the hyperbolic space H^2.

3. The hierarchy of L-functions associated with QCC is restricted by the finite-dimensionality of the Galois representation. Although in principle the classical space-time surface contains an infinite amount of potentially representable algebraic information, only a small part of it is represented in terms of quantum states.

2.3 About the p-adic variants of L-functions in the TGD framework

In the TGD framework, the existence of p-adic variants of L-functions and modular forms would be highly desirable. The conjecture that the roots of L-functions are algebraic numbers raises the hope that one could define these functions for p-adic integers s satisfying $s = O(p)$.

A stronger hypothesis is that L-functions are analogous to rational polynomials. The strongest meaning of this statement is that their values for rationals are rational. In particular the values of $\zeta(n)$ and $\xi(n)$ should be rational numbers. They are not. A weaker statement would be that the roots of L-functions are algebraic numbers.

The Hadamard product for ξ could make sense p-adically if the sums over the monomials defined by the products of the terms $(s_k\overline{s}_k)^{-1}) = 1/(1/4 + y_k^2)$, define algebraic numbers in the extension of rationals.

2.3.1 Kubota-Leopoldt variant of Dirichlet L-function

There exists proposals for the definitions of p-adic L-functions L_p (https://cutt.ly/wBgafkz). Both their domain and target are p-adic. The Kubota-Leopoldt variant $L_p(s, \chi)$ of Dirichlet L-function $L_p(s, \chi)$ serves as an example.

One starts from Dirichet L-function

$$L(s,\chi_m) = \sum_n \frac{\chi_m(n)}{n^{-s}} = \prod_p \frac{1}{1-\chi_m(p)p^{-s}} \quad , \tag{2.1}$$

where one has product over primes. $\chi_m(n)$ is Dirichlet character mod integer m (https://cutt.ly/WBKCpZZ), which satisfies $\chi_m(ab) = \chi_m(a)\chi_m(b)$ and vanishes if n is divisible by m. One restricts the consideration to negative integers $s = 1-n$. The factor $p^{-s} = p^{n-1}$ approaches zero in the p-adic sense for $n \to \infty$. Unexpectedly, just this Euler factor must be dropped from ζ.

One can express Dirichlet L-function in terms of generalized Bernoulli numbers (https://cutt.ly/DBgajxq) as

$$L(1-n,\chi_m) = -\frac{B_{n,\chi_m}}{n} \quad , \tag{2.2}$$

where $B_{n,\chi}$ is a generalized Bernoulli number defined by

$$\sum_{n=0}^{\infty} B_{n,\chi_m}\frac{t^n}{n!} = \sum_{a=1}^{f} \frac{\chi_m(a)te^{at}}{e^{ft}-1} \tag{2.3}$$

for χ_m a Dirichlet character (https://cutt.ly/gBgnhoh) with conductor f defined as the smallest power of prime for which χ_m is periodic.

The idea of the continuation is that Bernoulli numbers $B_n = -n\zeta(1-n)$, as also generalized Bernoulli numbers, are rational numbers and therefore make sense p-adically.

The Kubota-Leopoldt p-adic L-function $L_p(s,\chi)$ interpolates the Dirichlet L-function with the Euler factor associated with p removed. For positive integers n divisible by $p-1$, one has

$$L_p(1-n,\chi_m) = (1-\chi_m(p)p^{n-1})L(1-n,\chi_m) \quad . \tag{2.4}$$

When n is not divisible by $p-1$, this does not usually hold but one has

$$L_p(1-n,\chi_m) = (1-\chi_m\omega_p(-n)p^{n-1})L(1-n,\chi_m(n)\omega_p(n)) \quad . \tag{2.5}$$

Here ω is so called Teichmüller character $\omega(n)$, which is the p:th root of p-adic numbers n (https://cutt.ly/WBgabxC).

To my layman understanding, this definition depends on the interpretation of $1-n$ as an ordinary integer. For a p-adic integer, the sign does not have a real meaning so that this definition should make sense also for positive real integers interpreted as p-adic integers so that one can write $1-n = (1-(p-1)/(1-p)n = (1+(p-1)\sum_{k=0}^{\infty} p^k)n = (p+\sum_{k>0}^{\infty} p^k)$. Note that $1-n$ is p-adically of order $O(p)$, which suggests that quite generally this must be the case for the argument of ζ_p.

2.3.2 What could the p-adic variant of a function $f(x)$ mean?

It is not obvious what p-adicization of function $f(x)$ could mean. One can start from a Taylor expansion $f(x) = \sum f_n x^n$. A natural condition is that both the real and p-adic variant converge with an appropriate conditions on the norm of the argument used.

1. The naive approach requires that the coefficients f_n are identical. If algebraic numbers appear in coefficients f_n, an extension of rationals inducing that of p-adic numbers is needed.

   One could replace x with pinary expansion $x = \sum_n x_n p^n$, say identical rational numbers. For instance, for exponent function this would mean that the p-adic variant of $exp(x)$ exists only for $x_p < 1$. Typically, the p-adic expansion in powers p gives an infinite result in the real sense. One could argue that the correspondence must be more physical.

2. A physical correspondence is achieved in p-adic mass calculations [K5] by canonical identification, whose simplest variant is

$$I : x = \sum x_n p^n \rightarrow I(x) = \sum x_n p^{-n} \tag{2.6}$$

mapping p-adic numbers to real numbers. I is continuous and 2-1 for rationals since rationals in real sense have to equivalent expansions as real numbers since one has $1 = (p-1)/p)(1 + 1/p + 1/p^2 + ...)$ implying that the in inverse of I is 2-valued: $1_R \rightarrow 1$ and $1_R = (p - 1)/p)(1 + 1/p + 1/p^2 + ...) \rightarrow (p-1)p(1 + p + p^2 + ...))$ (for decimal expansions one has $1.000... = 0.99999...$).

3. For rational coefficients f_n, the simplest correspondence means reinterpretation as a p-adic number r_n/s_n. This would mean that small real values proportional to $1/p^{-n}$ are mapped to values with a large p-adic norm. A way avoid this is canonical identification. One can separate from rational valued f_n power p^k of p and map it to p^{-k} and treate the remaining factor as a p-adic number.

4. One can hope that this generalizes to the case when the coefficients f_n are in an extension of rationals defining extension of p-adic numbers and even in a possibly existing infinite-D extension fo rationals associated with f.

2.3.3 p-Adic Riemann zeta from Hadamard product

p-Adic Riemann zeta function could be obtained from Hadamard product if the roots of zeta are algebraic numbers.

1. The Hadamard product representation of $\zeta(s)$ (see `https://cutt.ly/ABgaQwE` and `https://cutt.ly/BBgaTf6`) is given by

$$\zeta(s) = \frac{e^{[(ln(2\pi)-1-\gamma/2)s]}}{2(s-1)\Gamma(1+s/2)} \prod_\rho (1 - \frac{s}{\rho})e^{s/\rho} \quad . \tag{2.7}$$

   Here γ is the Euler-Mascheroni constant and $\Gamma(s)$ is the Gamma function.

2. The roots $s = -2m$, $m > 0$ represent the first problem. The roots with $m = O(p)$ can have an arbitrarily small p-adic norm so that the product of the factors $1 - s/\rho$ from the negative real axis does not converge. Therefore one must drop these roots. This corresponds to the dropping of the Euler factor $1/(1-p^{-s})$ from the product form of ζ necessary in the definition of p-adic zeta by Kubota and Leopoldt. Note that this problem is not involved with the ξ function for which the expression of ξ reduces to $\xi(s) = (1 - s/\rho_k)(1 - s/\overline{\rho}_k)$.

3. Suppose that $s = O(p)$ holds true and the roots ρ of the ζ function are algebraic numbers. RH implies that they have modulus 1. Therefore one can expand $e^{s/\rho}$ in Taylor series and the factors $(1 - \frac{s}{\rho})e^{s/\rho})$ as ratios of the Taylor series to the first Taylor polynomial are of form $1 + O(p^2)$ so that the product converges.

   The factors $1/(\Gamma(1 + s/2)$ and $(1/(s-1)$ can be expanded around $s = 1$ to a convergent Taylor series.

4. The problematic term is the factor $e^{[(ln(2\pi)-1-\gamma/2]s]}$. If the coefficient $ln(2\pi) - 1 - \gamma/2$ is an algebraic number in the extension defined by the roots of zeta then also this exponent converges for p-adic integers $s = O(p)$, which belong to the extension of p-adic numbers conjectured to induced by the extension of rational defined by ζ. The existence of the Kubota-Leopoldt variant of the p-adic zeta indeed suggests that this is the case. If this is not the case, only $\xi(s)$ remains under consideration unless one allows transcendental extensions.

2.3.4 p-Adic ξ function from Hadamard product

ξ function (https://cutt.ly/5BjcCcv) is closely related to ζ and is much simpler. In particular, it lacks the trivial zeros forcing to drop from ζ the Euler factor to get ζ_p. ξ has a very simple representation completely analogous to that for polynomials (`https://cutt.ly/BBgaTf6`):

$$\xi(s) = \frac{1}{2}\prod_k (1-\frac{s}{s_k})(1-\frac{s}{\overline{s}_k}) \quad . \tag{2.8}$$

Only the non-trivial zeros appear in the product.

1. For $s = O(p)$, this product is finite but need not converge to a well-defined p-adic number in the infinite extension of p-adic numbers. Also the values of $\xi(s)$ at integer points are known to be transcendental so that the interpretation as a generalization of a rational polynomial fails. Note that the presence of an infinite number terms in the product can cause transcendentality of the coefficients of $\xi(s)$. Algebraic numbers are required. $\xi(2n)$ is proportional to π^2 and $\xi(2n+1)$ to $\zeta(2n+1)/\pi^2$. The presence of an infinite number of terms in the expansion of $\xi(s)$ can however cause this.

2. The Hadamard product can be written in the form

$$\xi(s) = \tfrac{1}{2}\prod_k (1+\tfrac{s(s-1)}{X_k}) \quad , \quad X_k = s_k\overline{s}_k \quad , \tag{2.9}$$

   in which the $s \leftrightarrow 1-s$ symmetry is manifest. The power series of $\xi(s) = \tilde{\xi}(u) = \sum a_n u^n$, $u = s(s-1)$, should converge for all primes p.

   If regards $s(s-1)$ as p-adic number and apply the inverse of I $s(s-1)$ to get real number.

   If the coefficients a_n of the powers series $\sum_n a_n u^n$ are numbers in an extension of rationals (not necessarily algebraic), the power series in s converges for $s = O(p)$ under rather mild conditioons. For instance, the coefficient of the zeroth order term is 1/2. The coefficient of the first order term in u is $-(1/2)\sum_k 1(s_k\overline{s}_k^{-1} = -2\sum_k (1+4y_k^2)^{-1}$.

One can deduce formal expressions for the Taylor coefficiens of $\xi(s)$.

1. Taking $u = s(s-1)$ to be the variable, the coefficients of u^n in $\xi(s) = \tilde{\xi}(u)$ are given by

$$\begin{array}{l} \sum_{U_n}\prod_{k\in U_n}\frac{1}{X_k} \quad , \\ X_k = s_k\overline{s}_k \quad . \end{array} \tag{2.10}$$

2. The calculation of the coefficients c_n is simple. In particular, c_1 and c_2 can written as

$$\begin{array}{l} c_1 = \frac{1}{2}\sum_i \frac{1}{X_i} \quad , \\ c_2 = \frac{1}{2}\sum_{i\neq j}\frac{1}{X_iX_j} \\ = \frac{1}{2}\sum_{i,j}\frac{1}{X_i}\frac{1}{X_j} - \frac{1}{2}\sum_i \frac{1}{X_i^2} \\ = \frac{1}{2}c_1^2 - \frac{1}{2}\sum_i \frac{1}{X_i^2} \quad . \end{array} \tag{2.11}$$

   The calculation reduces to the calculation of sums $\sum_1 /X_i k$, $k = 1, 2$.

3. Also the higher coefficients c_n can be calculated in a similar way recursively by subtracting from the sum $\sum_{i_1...i_n}\prod_{i_k} X_{i_1}^{-1} = c_1^n$ without the constraint $p_i \neq p_j \neq ...$ the sums for which $2, 3, ..., n$ primes are identical. One obtains a sum over all partitions of U_n. A given partition $\{i_1, ..., i_k\}$ contributes to the sum the term

$$d_{i_1,...,i_k}\prod_{l=1}^k c_{i_l} \quad , \quad \sum_{i=1}^k n_i = n \quad . \tag{2.12}$$

The coefficient $d_{i_1,...,i_k}$ tells the number of different partitions with same numbers $i_1,...,i_k$ of elements, such that the n_i elements of the subset correspond to the same prime so that this subset gives c_{n_i}. Note that the same value of i can appear several times in $\{i_1,...,i_k\}$.

The outcome is that the expressions of c_n reduce to the calculation of the numbers $A_k = \sum_i 1/X_i^k$.

2.3.5 Could one deduce conditions on the coefficients of ξ from number theoretical democracy?

Can one pose additional conditions in the case of ζ or ξ? I have difficulties in avoiding a tendency to bring in some number theoretic mysticism in hope say something interesting of the values of the coefficient X_n in the power series $\xi = c_n u_n$, $u = s(s-1)$, which can calculated from the Hadamard product representation. Number theoretical democracy between p-adic number fields defines one form of mysticism.

There is however also a real problem involved. There is a highly non-trivial problem involved. One can estimate the real coefficients X_k only as a rational approximation since infinite sums of powers of $1/X_k$ are involved. The p-adic norm of the approximation is very sensitive to the approximation.

Therefore it seems that one *must* pose additional conditions and the conditions should be such that the coefficients are mapped to numbers in extension of p-adic numbers by the inverse of I as such so that they should be algebraic numbers or even transcendentals in a finite-D transcendental extension of rationals, if such exists.

1. One could argue that the coefficients c_n must obey a number theoretical democracy, which would mean that they can distinguish p-adically only between the set of primes p_k appearing as divisors of n and the remaining primes. One could require that c_n is a number in a finite-D extension of rationals involving only rational primes dividing n.

2. One could pose an even stronger condition: the coefficients c_n must belong to an n-D algebraic extension of rationals and thus be determined by a polynomial of degree n. Polynomials P of rational coefficients p_n bring in failure of the number theoretic democracy unless one has $p_n \in \{0, \pm 1\}$. For $p = 2$ one does not obtain algebraic numbers. For $p = 3$ this would bring in $\sqrt{5}$.

3. These conditions would guarantee that for a given prime p the coefficients of the expansion would be unaffected by the canonical identification I and at the limit $p \to \infty$ the Taylor coefficients of p-adic ξ_p would be identical with those of ξ.

4. One could allow finite-D transcendental extensions of p-adic numbers. These exist. Since e^p is an ordinary p-adic number, there is an infinite number of extensions with a basis given by the powers roots $e^{k/n}$, $k = 1, ..., np-1$ define a finite-D transcendental extension of p-adics for every prime p.

   The strongest hypothesis is that the coefficients c_k are expressible solely as polynomials of this kind of extensions with coefficients, which are algebraic numbers of integers in an extension of rationals by a k:th order polynomial P_k, whose coefficients belong to $\{0, \pm 1\}$.

This picture suggests a connection with the hyperbolic geometry H^2 of the upper half-plane, which is associated with ζ and ξ via Langlands correspondence.

1. The simplest option is that the roots of P_k correspond to the k:th roots x_i of unity satisfying $x_i^k = 1$ so that $cos(n2\pi/k)$ and $sin(n2\pi/k)$ would appear as coefficients in the expression of c_k. The numbers $e^{k/n}$ would be hyperbolic counterparts for the roots of unity.

2. The coefficients c_k would be of form

$$c_k = \sum_{i,j} c_{k,ij} e^{i/k} exp(\sqrt{-1} 2\pi(j/n)) \quad , \quad c_{k,rs} \in \{0, \pm 1\} \quad . \tag{2.13}$$

The coefficients could be seen as Mellin-Fourier transforms of functions defined in a discretized hyperbolic space H^2 defined by 2-D mass shell such with coordinates $(cosh(\eta), sinh(\eta)cos(phi), sinh(\eta)sin(\phi))$ $\eta) = i/k$, $\phi = 2\pi j/n$. η is the hyperbolic angle defining the Lorentz boost to get the momentum from rest momentum and ϕ defines the direction of space-like part of the momentum. Upper complex plane defines another representation of H^2. The values of functions are in the set $\{0, \pm 1\}$.

3. The points of H^2 associated with a particular c_k would correspond to the orbit of a discrete subgroup of $SO(1,1) \times SO(2) \subset SO(1,2) \subset SL(2,R)$ ($SL(2,R)$ is the covering of $SO(1,2)$).

   A good guess is that this discretization could be regarded as a tessellation of H^2 and whether other tessellations (there exists an infinite number of them corresponding to discrete subgroups of $SL(2,R)$ could be associated with other L-functions. Mellin transform relates Jacobi theta function (`https://cutt.ly/1B96SSE`), which is a modular form, to $2\xi/s(s-1)$. Therefore $SL(2,C)$, having $SL(2,R)$ as subgroup acting as isometries of H^2, is the appropriate group.

   Note that the modular forms associated with the representations of algebraic subgroups of $SL(2,C)$ defined by finite algebraic extensions of rationals correspond to L-functions analogous to ζ. Now one would have a hyperbolic extension of rationals inducing a finite-D extension of p-adic numbers.

Just for curiosity and to see how the proposal could fail, one can look at what happens for the first coefficient c_1 in $\xi(s) = \tilde{\xi}(s(s-1)) = \sum c_n s^n$.

1. c_1 would be exceptional since it cannot depend on any prime. c_2 could involve only $p = 2$, and so on.

2. The only way out of the problem is to allow finite-D transcendental extensions of p-adic numbers. These exist. Since e^p is an ordinary p-adic number, there is an infinite number of extensions with a basis given by the powers roots $e^{k/n}$, $k = 1, ..., np-1$ define a finite-D transcendental extension of p-adics for every prime p. For ξ the extension by roots of unity could be infinite-dimensional.

   The roots $e^{k/n}$, $k \in 1, ...n$ belong to this extension for all primes p and are in this sense universal. One can construct from the powers of $e^{k/n}$ expressions for c_1 as $c_1 = \sum_k a_k e^{-k/n}$, $a_k \in \{\pm = 0, \pm 1\}$.

3. This would allow to get estimates for n using $x_1 = d\xi/ds(0) \simeq .011547854 = 2c_1$ as an input:

$$c_1 = \sum a_k e^{-k/n} = \frac{x_1}{2} \quad .$$

   For instance, the approximation $c_n = e - e^{(n-1)/n}$ would give a rough starting point approximation $n \sim 117$. It is of course far from clear whether a reasonably finite value of n can reproduce the approximate value of c_1.

2.4 What about the p-adic variants of modular forms?

What about modular forms as analogs of partition functions? Also they should exist for the same value range for integer conformal weights.

1. Very roughly, L-function is obtained from the Fourier expansion of modular forms

$$Z(s) = \sum c_n q^n \quad , \quad q(s) = e^{i2\pi ns} \tag{2.14}$$

   by the replacement

$$q^n \to n^{-s} \quad . \tag{2.15}$$

2. A natural condition is that the p-adic variants of $Z(s)$ and $L(s)$ converge for the same range of values of s. The appearance of $i2\pi$ in the exponential is problematic from the point of view of p-adicization.

3. In the p-adic thermodynamics modular form corresponds to a partition function and the natural identification of q is as

$$q = p^{n/T_p} \quad , \tag{2.16}$$

where n is conformal weight as eigenvalue $h = n$ of the scaling generator L_0 representing mass squared value and $T_p = 1/k$ is the p-adic temperature. n is interpreted as a p-adic integer so that the partition function converges extremely rapidly in p-adic mass calculations for which p is very large for elementary particles ($= M^{127} = 2^{127} - 1$ for electron).

Note that ordinary Boltzmann weights $exp(-n/T_p)$ would make sense if $1/T_p = O(p)$ holds true. The sum over Boltzmann weights would not however converge since $exp(-n/T_p)$ would have p-adic norm equal to 1. Therefore one must replace e by p: in the real context this would mean only a redefinition of temperature.

4. Naively, the correspondence between modular forms and L-functions should be $q = p^{n/T_p}" = "e^{nln(p)/T} \rightarrow n^s$, $s = O(p)$, by using the definition $\zeta = \sum n^{-s}$. This would suggest the correspondence $1/T_p = k \rightarrow s$. This would conform with the interpretation as p-adic integers but why should one have $k = O(p)$ as required by the definition based on the Hadamard formula? Should one simply assume that $T_p = k \rightarrow s/p$?

Can one make sense of the summand n^{-s}?

1. If n is of form $n = 1 + O(p)$, p-adic logarithm $log_p(n) = log(1 + O(p))$ exists as Taylor series and is of order $O(p)$ and the exponent $exp(log(n)s)$ exists even for $s = O(1)$.

2. p-Adic logarithm can be defined for $p \geq k \geq 0$ by using the finite field property of p-adic integers $0 < x < p$. In this case $log(n)$ contains also an $O(1)$ term so that n^{-s} would make sense only for $s = O(p)$. Therefore there would be a consistency between two definitions for integers n not divisible by p. For $n \propto p^n$ one must have an extension allowing $log(p)$. Should the extension of rationals possibly assignable to zeta contain also logarithms of primes, which are not algebraic numbers?

3. An alternative way is to drop integers n proportional to powers of p from from the definition of ζ. This corresponds to the dropping of the Euler factor $1/(2 - p^{-s})$ associated with p in the product form of zeta used to define zeta for negative integers.

4. One could also restrict the consideration to ξ and use the Hadamard product.

2.5 p-Adic thermodynamics and thermal zeta function

The Dirichlet series defines an L-function. The definition of Dirichlet series is following. Consider entities a with integral weight $w(a)$, say quantum states characterized by conformal weight n. Suppose that there are $g(n)$ states with conformal weight n. The sum $\sum w(a)^{-s} = \sum g(n)n^{-s}$ defines the Dirichlet series with nice properties.

This kind of system also has a description in terms of a partition function, which assigns to the partition function an analog of modular form. In the assignment of an L-function to a modular form, the $\sum g(n)exp(-n/T)$ is replaced with $\sum g(n)n^{-s}$ in the real case.

In the p-adic case $\sum g(n)p^{n/T_p)}$ is replaced with a similar sum. The p-adic temperature T_p is quantized to $T_p = 1/n$ for the p-adic partition function. In the p-adic case, the number theoretical existence allows only integer values of $1/T_p$ as a counterpart of s. One can also consider finite-D extensions of rationals for which p-adic extension allows some p-adic roots of integers.

If the p-adic partition function Z for the scaling generator L_0 appearing in the p-adic mass calculations [K5, K2], allows an analog of the zeta function and if it satisfies RH hypothesis, one obtains conformal confinement in weak and strong form and if the roots of the L-function are algebraic numbers, also Galois confinement. This could define a 4-D space-time surface as a classical correlate of the thermal state or its complex square root.

2.6 Could elementary particle vacuum functionals define analogs of L-functions?

Elementary particle vacuum functionals (EPVFs) [K2] are defined in the space of conformal equivalences of partonic 2-surfaces and therefore correspond to wave functions in WCW. A partonic 2-surface with a given topology allows a complex structure and moduli space for them. The induced metric defines the conformal equivalence class. Teichmueller space parameterizes this moduli space and is part of WCW. Explanation of the family replication phenomenon is based on hyper-ellipticity.

EPVFs are identified as modular invariant modular forms and are constructed from Jacobi theta functions, which for a given genus g depends on $D = 3g - 3$ Teichmueller parameters forming a complex symmetric matrix with positive imaginary part for $g \geq 2$ and on $D = 0$ *resp.* $D = 1$ parameters for g_0 *resp.* $g = 1$. This space can be regarded as a generalization of the upper half of the complex plane (hyperbolic space H^2). For $g = 1$ EPVFs depend on a single theta parameter and the corresponding L-function would satisfy RH.

One can assign to these modular forms L-functions by developing them to Fourier series as $\sum_n c_n q^n$, $q = exp(i2\pi s)$. To this series one can assign an L-function by the replacement $q^n \rightarrow s^{-n}$. I am not quite sure how closely this corresponds to Mellin transform (`https://cutt.ly/NBgnluF` and `https://cutt.ly/dBgncAR`).

The general philosophy described above suggests that it should be possible to assign to EPVF an L-function of a single variable, whose roots would define a space-time surface providing classical representation of the quantum state considered. One should define a multivariable L-function as an analog of poly-zeta and assign to it an L-function of a single variable.

1. One can define multivariable analogs of L-functions. One can imagine a straight forward generalization of the definition of L-function by starting from a multiple Fourier series of Riemann theta function with respect to its arguments, which are Teichmüller parameters Ω_{ij} parameterizing conformal equivalence classes of partonic 2-surfaces. One has $\Omega_{ij} = \Omega_{ji}$, $Im(\Omega_{ij}) > 0$ (one has a higher-D analog of the upper half-plane). The variables s_k are in 1-1 correspondence with the variables Ω_{ij}, $j \geq i$.

   The analogs of L-functions depending on several complex variables $s_1, ..., s_n$ cannot be as such used as a generalization of polynomials. One should identify an L-function of a single variable. One should get rid of the variables $s_2, ..., s_n$.

2. Could one mimic the construction of twistor amplitudes? Could one solve first a residue of a pole of generalized L-function with respect to s_n as a function of $s_1, s_2, ..., s_{n-1}$, after that the residue of the pole with respect s_{n-1} and so on .... At the final step one would get a polynomial of a single variable s_1. Could it be analogous to an L-function of a single variable and have zeros with half-integer valued real part?

   The interpretation would be as a residue integral over variables $s_2, ..., s_n$: similar integrals appear in the construction of twistor amplitudes. There is evidence that his idea might work for ξ functions (`https://cutt.ly/jBgnmOJ`). On the theory of normalized Shintani L-function and its application to Hecke L-function see (`https://cutt.ly/SBgnTUN`).

The following argument provides support for this idea in the case of multiple zeta functions (polyzetas) (see `https://cutt.ly/oBgnO54`, `https://cutt.ly/cBgnXDn` and `https://cutt.ly/ZBgnV6Y`).

1. Poly-zetas have $\{s_1, s_2, ..., s_n\}$ as arguments. One has $\zeta(s_1, ..., s_n) = \sum_{n_1 > n_2 > ... > n_k > 0} \prod_{i=1}^k s_i^{-n_i}$. Otherwise one would have a product of ordinary zeta functions.

2. In the Wikipedia article, a variant of polyzeta denoted by $S(s_1, ..., s_n)$ is introduced as $S(s_1, ..., s_n) = \sum_{n_1 \geq n_2 \geq ... \geq n_k > 0} \prod_{i=1}^k s_i^{-n_i}$: ">" is replaced with "$\geq$" in the summation.

   By separating from the sum various cases in which 2 or more integers n_i are identical, one can decompose $S(s_1, ..., s_n)$ to a sum over products of the ordinary zeta functions with arguments, which are sums $s_i + s_{i+1} + s_{i+r}$ of subsequent arguments associated with partitions of $\{s_1, ..., s_n\}$ to l subsets $\{s_1, s_2, ..., s_{k_1-1}\}$, $\{s_{k_1}, ..., s_{k_2-1}\}$, ..., $\{s_{k_l}, ..., s_n\}$ respecting the ordering. One can think that the arguments s_i are along a line, and divide the line in all possible ways to segments.

3. One can also form a symmetrized sum $\sum_{\Pi}\zeta(s_{\Pi(1)},...,s_{\Pi(k)})$ of $\zeta(s_1,...,s_k)$ over permutations of $\{s_1,s_2,...,s_k\}$ to l subsets. The theorem of Hoffmann, mentioned in the Wikipedia article, states that the symmetrized polyzeta reduces to a sum over products of ordinary zetas assigned over all partitions such that the argument associated with a given subset of partition is the sum $s_{i_1}+...+s_{i_r}$.

4. If the multiple L-function corresponds to the symmetrized variant of $\zeta(s_1,...,s_n)$, its $k-1$-fold residue integral decomposes to a sum of residue integrals, which give a vanishing result except in the case of $\zeta(s_1)\times...\times\zeta(s_k)$ of k zetas assignable to the maximal partition.

   If one assumes $s_1+s_2+...+s_k=s$, for the multiple residue integration contour, the integral is proportional $\zeta(s-k)$. The non-trivial zeros are at the critical line $Re(s)=k+1/2$ and the trivial zeros are at the points $s=k-2m$, $m\geq 0$. The permutation symmetry of the multiple residue integral suggests that the symmetrization can be performed by using symmetry of the integration measure so that also in this case the outcome is proportional to $\zeta(s-k)$.

2.7 Could the tessellations of H^3 be obtained from those of H^2 by holography?

A rather attractive idea is that the 2-D modular forms in 2-D hyperbolic manifolds of H^2 allow a holographic continuation to 3-D modular forms in 3-D hyperbolic manifolds of the mass shell H^3.

1. Compactification of a modular curve is determined by the infinite subgroup Γ of $SL(2,R)$ in the hyperbolic plane H^2 is obtained by adding cusp points located at real axis. The hyperbolic unit cell as hyperbolic 2-manifold has cusps as sharp tips.

2. Does the 2-D hyperbolic manifold extend to a hyperbolic manifold in H^3 having $SL(2,C)$ as a covering group of isometry group $SO(1,3)$? Modular function in H^3. The modular curve as 2-D hyperbolic manifold would be extended to a hyperbolic 3-manifold (`https://cutt.ly/NBgbAYC`) and would have a 2-D hyperbolic manifold as its boundary just like H^2 has real line as a boundary.

   Hyperbolic 3-manifold could be identified as a 3-surface at H^3 defining the unit cell of tessellation. Compactication would add points to the counterparts of cusps as singular points, which would naturally correspond to the boundary of the coset space forming a 2-D hyperbolic manifold.

   The continuation from H^2 to H^3 would correspond to the extension of γ as a subgroup of $SL(2,R)$ to its complexification as a subgroup of $SL(2,C)$. The extension would be analogous to the continuation of real analytic function to complex analytic function as a form of holography.

3. The physical analogy with the boundary of Fermi torus [L9] is rather obvious. This would conform with the strong form of holography stating that the boundary of 3-surface determines the 3-surface proposed to apply at the light-like boundary of CD. The holography would be however restricted to the mass shells H^3 determined as root of a polynomial and possibly even L-function. An interesting question is whether X^2 fixes also its 3-D light-like orbit by holography. Quantum criticality suggests a failure of a strict determinism.

2.8 About the identification of L-group

How could one understand in the TGD framework, the L-group, or ^LG, as a Langlands dual? The standard approach is described in `https://cutt.ly/iBgnMIF`. Langlands dual ^{L}G and L-group are more or less the same. L-group is a semidirect product of $^LG^0$ and Galois group such that the Galois group has natural action in the matrix representation of the algebraic group G with matrix elements. This is the case if G is defined over a field containing the extension of rationals to which the Galois group is associated. Algebraic groups over global fields (extensions of rationals) can be regarded as analogs of Lie groups and the Dynkin diagrams assignable to Lie algebras appear in their classification.

The guess based on TGD vision was following. One assumes global field that is a finite extension of rationals. Lorentz group, $SL(2,C)$, etc. are discretized.

1. In TGD picture, Galois group permutes mass shells. The isotropy group acts on momentum components but keeps them on mass shell. Lorentz group mixes momentum components. Can one form a larger group from these groups by forming the products of group elements.

2. A free group from from G_1 and G_2 with amalgamation is obtained by adding some relations by using a third group U inbedded to both groups by homomorphism (`https://cutt.ly/DBgn9Q2`). G_1 and G_2 are glued together along U.

   In the recent case, G_1 could corresponds to Galois group and G_2 to Lorentz group $SO(1,3)$ or its covering for a global field extension. U corresponds to a subgroup of Galois group and of Lorentz group. G_2 can correspond to the non-compact groups defined by the truncated Virasoro algebra or symplectic algebra of $\delta M^4 \times CP_2$. U must be a subgroup of Galois group leaving the root fo P defining mass squared invariant.

3. What about Galois singletness in this case? The group obtained in this way permutes mass shells. The automorphic forms in the extended group be invariant under Galois group or its amalgamated product with a discrete infinite subgroup of $SL(2,C)$.

4. The free product and amalgamated free product construction is extremely general. It could work even for an extension of finite field or extension of corresponding p-adic number field and $SL(2,C)$. Here unramified and ramified primes pop up. The induced Galois group looks more natural here.

What about quaternionic automorphisms, which is analog of Galois group? The amalgamated free product of (discrete subgroups of) quaternionic automorphisms with Galois group could be important. Free product with amalgamation would naturally apply to Galois group, quaternionic automorphisms, $SL(2,C)$ and subgroups of conformal transformations.

2.8.1 The identification of candidates for the reductive groups

Extension of irreducible representations of Galois group to representations of reductive groups extended by Galois group, so called L-group, are suggested by the Langlands program and in the TGD framework they would be very natural. These extensions could define WCW spinor fields. What candidates does TGD offer for the reductive groups in question?

1. In TGD, the infinite-D (super-)symplectic group assignable to $\delta M^4_+ \times CP_2$ defines a candidate for the isometries of WCW. The Lie algebra A of this group corresponds to Hamiltonians as functions defined in $\delta M^4_+ \times CP_2$. The basis of Hamiltonians can be assumed to be products of functions defined in δM^4_+ and CP_2. For δM^4_+ one has irreps of $SO(3)$ acting in δM^4_+ and proportional to a power of r^n of the light-like radial coordinates, where n is conformal weight. For CP_2 one has functions defining irreps of $SU(3)$.

2. The Lie-algebra A allows infinite fractal hierarchies formed by sub-algebras A_n with radial conformal weights coming as n-multiples of the conformal weights of the full algebra. The gauge conditions state that A_n and the commutator $[A_n, A]$ annihilate the physical states. These conditions generalize to other symmetry groups assignable to the light-like 3-surfaces defining partonic orbits and to the extended conformal transformations of the metrically 2-D light-cone δM^4_+.

   The first naive guess is that the gauge conditions effectively reduce the symplectic group to finite-D symplectic group $Sp(2m)$ or its reductive subgroup acting linearly. In this case one might have infinite-D representations

3. One can also consider the possibility that the gauge conditions for the radial conformal transformations are weakened to similar conditions as in the case of A. Similar conditions could apply to the algebras associated with the light-like 3-surfaces.

2.9 A comment on $M^8 - H$ duality in fermion degrees of freedom in relation to Langlands duality

Gary Ehlenberg sent an URL of a very interesting Quanta Magazine article, which discusses a work related to Langlands program and provides some rather concrete insights how $M^8 - H$ duality [L5, L6], relating the number theoretic and geometric views of TGD, could relate to the Langlands duality.

Langlands duality relates number theory and geometry. At the number theory side one has representations of Galois groups. On the geometry side one has automorphic forms associated with the representations of Lie groups. For instance, in coset spaces of hyperbolic 3-space H^3 in the case of the Lorentz group.

The work could be highly interesting from the TGD perspective. In TGD, the $M^8 - H$ duality generalizes momentum-position duality so that it applies to particles represented as 3-surfaces instead of points. $M^8 - H$ duality also relates physics as number theory and physics as geometry. Much like Langlands duality. The problem is to understand $M^8 - H$ duality as an analog of Langlands duality.

1. $H = M^4 \times CP_2$ is the counterpart of position space and particle corresponds to 3-surface in H. Physics as (differential) geometry applies at this side.

   The orbit of 3-surface is a 4-D space-time surface in H and holography, forced by 4-D general coordinate invariance, implies that space-time surfaces are minimal surfaces irrespective of the action (general coordinate invariant and determined by induced geometry) . They would obey 4-D generalization of holomorphy and this would imply universality.

   These minimal surfaces are also solutions of a nonlinear geometrized version of massless field equations. Field-particle duality has a geometrized variant: minimal surface represents in its interior massless field propagation and as an orbit of 3-D particles the generalization of a light-like geodesic. Hence a connection with electromagnetism mentioned in the popular article, actually metric and all gauge fields of the standard model are geometrized by induction procedure for geometry.

2. M^8, or rather its complexification M_c^8 (complexification is only with respect to Minkowski time) corresponds to momentum space and here the orbit of point-like particle in momentum space is replaced with a 4-surface in M^8, *oractuallyitscomplexification*M$_c^8$.

   The 3-D initial data for a given extension of rationals could correspond to a union of hyperbolic 3-manifolds as a union of fundamental regions for a tessellation of H^3 consistent with the extensions, a kind of hyperbolic crystal. These spaces relate closely to automorphic functions and L-functions.

   At the M^8 side polynomials with rational coefficients determine partially the 3-D data associated with number theoretical holography at M^8−*side.Thenumbertheoreticaldynamicalprinciplestatesthatth*
   *timesurfaceintheoctonionic*M$_c^8$ is associative and initial data correspond to 3-surfaces at mass shells $H_c^3 \subset M_c^4 \subset M_c^8$ determined by the roots of the polynomial.

3. $M^8 - H$ duality maps the 4-surfaces in M_c^8 to space-time surfaces in H. At the M^8 side one has polynomials. At the geometric H-side one has naturally the generalizations of periodic functions since Fourier analysis or its generalization is natural for massless fields which space-time surfaces geometrize. L-functions represent a typical example of generalized periodic functions. Are the space-time surfaces at H-side expressible in terms of modular function in H^3?

Here one must stop and take a breath. There are reasons to be very cautious! The proposed general exact solution of space-time surfaces as preferred extremals realizing almost exact holography as analogs of Bohr orbits of 3-D surfaces representing particles relies on a generalization of 2-D holomorphy to its 4-D analog. The 4-D generalization of holomorphic functions [L19] assignable to 4-surfaces in H do not correspond to modular forms in 3-D hyperbolic manifolds assignable to the fundamental regions of tessellations of hyperbolic 3-space H^3 (analogs of lattice cells in E^3).

Fermionic holography reduces the description of fermion states as wave functions at the mass shells of H^3 and their images in H under $M^8 - H$ duality, which are also hyperbolic 3-spaces.

1. This brings the modular forms of H^3 naturally into the picture. Single fermion states correspond to wave functions in H^3 instead of E^3 as in the standard framework replacing infinite-D representations of the Poincare group with those of $SL(2,C)$. The modular forms defining the wave functions inside the fundamental region of tessellation of H^3 are analogs of wave functions of a particle in a box satisfying periodic boundary conditions making the box effectively a torus. Now it is replaced with a hyperbolic 3-manifold. The periodicity conditions code invariance under a discrete subgroup $\Gamma \subset SL(2,C)$ and mean that $H^3 = SL(2,C)/U(2)$ is replaced with the double coset space $\Gamma \backslash SL(2,C)/U(2)$.

   Number theoretical vision makes this picture more precise and suggests ideas about the implications of the TGD counterpart of the Langlands duality.

2. Number theoretical approach restricts complex numbers to an extension of rationals. The complex numbers defining the elements $SL(2,C)$ and $U(2,C)$ matrices are replaced with matrices in discrete subgroups $SL(2,F)$ and $U(2,F)$, where F is the extension of rationals associated with the polynomial P defining the number theoretical holography in M^8 inducing holography in H by $M^8 - H$ duality. The group $\Gamma definingtheperiodicboundaryconditionsmustconsistofmatricesi$

3. The modular forms in H^3 as wave functions are labelled by parameters analogous to momenta in the case of E^3: in the case of E^3 they characterize infinite-D irreducible representations of $SL(2,C)$ as covering group of $SO(1,3)$ with partial waves labelled by angular momentum quantum numbers and spin and by the analog of angular momentum associated with the hyperbolic angle (known as rapidity in particle physics): infinitesimal Lorentz boost in the direction of spin axis.

   The irreps are characterized by the values of a complex valued Casimir element of $SL(2,C)$ quadratic in 3 generators of $SL(2,C)$ or equivalently by two real Casimir elements of $SO(1,3)$. Physical intuition encourages the shy question whether the second Casimir operator could correspond to the complex mass squared value defining the mass shell in M^8. It belongs to the extension of rationals considered as a root of P.

   The construction of the unitary irreps of $SL(2,C)$ is discussed in Wikipedia article. The representations are characterized by pairs of half-integer $j_0 = n/2$ and imaginary number $j_1 = i\nu$. Since the representations in question are H^3 analogs of the irreducible representations of Poincare group in M^4 with E^3 replacing H^3 the natural interpretation of j_0 would be as spin. The states of the representation would represent partial waves with definite value of j. In TGD, $j_0 = 1/2$ would be in a special role.

   The values of j_0 and j_1 must be restricted to the extension of rationals associated with the polynomial P defining the number theoretic holography.

4. The Galois group of the extension acts on these quantum numbers. Angular momentum quantum numbers are quantized already without number theory and are integers but the action on the hyperbolic momentum is of special interest. The spectrum of hyperbolic angular momentum must consist of a union of orbits of the Galois group and one obtains Galois multiplets. The Galois group generates from an irrep with a given value of j_1 a multiplet of irreps.

   A good guess is that the Galois action is central for $M^8 - H$ duality as a TGD analog of Langlands correspondence. The Galois group would act on the parameter space of modular forms in $\Gamma(2,F)/U(2,F)$, F and extension of complex rationals and give rise to multiplets formed by the irreps of $SL(2,F)$.

To sum up, $M^8 - H$ duality [L5, L6] is a rather precisely defined notion (I am of course using the standards of physicist).

1. At the M^8 side one has polynomials and roots and at the H-side one has automorphic functions in H^3 and "periods" are interpreted as quantum numbers. What came first in my mind was that understanding of M^8 duality boils down to the question about how the 4-surfaces given by number theoretical holography as associativity of normal space relate to those given by holography (that is generalized holomorphy) in H.

2. However, it seems that the problem should be posed in the fermionic sector. Indeed, above I have interpreted the problem as a challenge to understand what constraints the Galois

symmetry on M^8 side poses on the quantum numbers of fermionic wave functions in hyperbolic manifolds associated with H^3 and defined b the extension of rationals in question. I do not know how closely this problem relates to the problem that Ben-Zvi, Sakellaridis and Venkatesh, whose work is discussed in the popular article mentioned in the beginning, have been working with.

3 More about Langlands correspondence in the TGD framework

I listened to an extremely interesting interview with Edward Frenkel. Thanks to Marko Manninen for providing the link (see this). Frenkel explained more than two hours aspects of the number theoretic Langlands correspondence [A11, K3, A4, A3] in order to provide the background making it possible to get to the geometric Langland correspondence in the next interview! The fact of course is that one must learn the basic notions first.

If one should summarize the Langlands correspondence (LC) very briefly, one might say that LC can be seen as a kind of grand unification of mathematics. LC relates typically two totally different fields of mathematics. Modular forms is one aspect of the LC. Galois groups permuting the roots of polynomials are a second aspect, also mentioned in the interview.

3.1 Summary of Frenkel's interview

3.1.1 Polynomials defining elliptic surfaces in finite fields and modular forms in hyperbolic plane

The example related to the number theoretic LC discussed in the interview relates number theory and so-called modular forms defined in the hyperbolic plane H^2 (see). The graphics of Escher gives a good idea of what a hyperbolic plane looks like.

1. The modular forms known as elliptic functions are doubly periodic functions in complex plane and serve analogs of plane waves appearing as solutions of field equations of free field theories.

2. Lorentz group $SL(2,C)$ acts as global conformal symmetries of the complex plane compactified to the Riemann sphere CP_1. $SL(2,C)$ consists of 2×2 matrices with unit determinant and acts linearly on 2-spinors and as Möbius transformations on the points of the complex plane CP_1, whose points correspond to the ratios z^1/z^2 of the complex spinor components. 2-D hyperbolic space H^2 can be represented as upper half-plane of the complex plane or as an interior of a unit disk and has the real Lorentz group $SL(2,R)$ as conformal symmetries mapping the real axis or the boundary of the disk to itself.

   For H^2, the modular group can be identified as some subgroup $SL(2,Z)$, having integer valued matrix elements satisfying some additional conditions. Modular forms have the modular group as their symmetries. The modular subgroup can correspond also to a group $SL(2,Z^2)$ where Z^2 consist of Gaussian integers n_1+in_2. Modular transformations act as translations in the complex plane. There is an infinite number of analogs of modular groups since the elements could also be algebraic integers Z_E in some extension E of rationals.

The example discussed in the interview was very inspiring also from the TGD point of view.

1. The example was about the number of solutions to certain kinds of Diophantine equations defining roots of polynomials involving 2 variables x,y with integer coefficients in finite field defined by integers modulo prime p. Modulo p corresponds to a generalization of the clock arithmetics in which 13=1, 14=2, etc.: now there would be p hours per day. The polynomials considered are cubic polynomials, which third order polynomials of x and second order polynomials y satisfying some additional conditions, which guarantee that they allow a modular symmetry analogous to a 2-D translation symmetry.

   The independent variables x,y can be numbers in finite fields, integers, reals, complex numbers. In other words, the equations are number-theoretically universal. This is what makes algebraic geometry so beautiful.

2. If the variables x, y are complex numbers, the solutions are 2-D surfaces in 2-D complex space C^2 having complex coordinates x and y. With some additional assumptions about the integer coefficients they are elliptic curves, which have the topology of the torus. Elliptic curve, or a general cubic curve, can be written in rather general case in the canonical form (see this):

$$y^3 = x^2 + ax + b \ . \tag{3.1}$$

3. Elliptic curve intersects line (in complex case complex line, that is plane) in 3 points except in the situations, when the equation allows degenerate roots. This reflects the fact that the equation has 3 roots. A similar equation appears in the cusp catastrophe. The coefficients are however real numbers rather than integers in this case. The 4-D generalization of cusp catastrophe is very interesting also in the TGD framework,

4. The elliptic curve can be parameterized in terms of elliptic functions, which can be regarded as doubly periodic functions in the complex plane. They are non-linear analogs of plane waves. The rational points of an elliptic curve form a group which is finitely generated. Elliptic curves are modular, which means that it can be constructed as a quotient of a complex plane under modular grou, which is a subgroup of the group $SL(2, Z)$ analogous to a discrete subgroup of translational group but acting on H^2 rather than Euclidean space E^2.

5. With this background one can consider a general equation defining elliptic curve and restrict the solutions to integers and consider solutions x,y as integers modulo p.

   What is the number of solutions for a given prime p. This is the problem. Langland's discovery implies that one can find a modular form in the hyperbolic plane H^2, whose Taylor expansion contains terms with coefficients, which code for the numbers of the solutions for any prime p: the problem of finding the number of solutions is solved for almost all primes at the same time! There exists a finite number of exceptions but still this is quite an impressive achievement.

   Anyone can write a computer code listing the numbers of solutions for primes $p = 2, 3, 5, 7, ...$, This is a gigantic leap in understanding and LC generalizes this. There is also a geometric variant of this correspondence.

3.1.2 Galois group, dual group and number theoretic LC

Galois groups define number theoretic symmetries. Consider a polynomial $P(x)$ of single variable x with coefficients, which are algebraic integers in some extension E of rationals. The standard definition for the Galois group is as a group acting in the extension of E as permutations of the roots acting trivially in E.

Let G be a reductive Lie group G represented in an extension K of field k, which has Galois group $Gal(K/k)$. LC relates the "good" irreducible representations G to the representations of Langlands dual ${}^LG$, which has the same root datum as G and is therefore infinitesimally equivalent with G. ${}^LG$ can be regarded as the semidirect product ${}^LG^o \ltimes Gal(K/k)$, where ${}^LG^o$ is the connected component of ${}^LG$. The number of connected components of ${}^LG$ corresponds to the order of the Galois group, which implies that ${}^LG$ depends on K via its Galois group.

LC states roughly that there is one-one correspondence between "good" representations of G and homomorphisms of the Galois group $Gal(K, k)$ to the Langlands dual ${}^LG$. One could say that the extension by the Galois group gives rise to a kind of number theoretic degree of freedom and that the irreducible representations of the Galois group become additional degrees of freedom.

3.1.3 LC and TGD

It must be emphasized that LC is extremely general and the forms of LC discussed in this article from the TGD perspective are not expected to be the only ones realized in TGD.

1. In the electric-magnetic duality (Montonen-Olive duality) of gauge theories [B1], electric and magnetic couplings are combined to a complex number $z = z_1/z_2$ representing a point of CP_1. Electric-magnetic duality corresponds to S-duality as the Möbius transformation $z \to -1/z$.

   More generally, $SL(2,Z)$ transformations act as Möbius transformations induced by the linear action on (z_1, z_2) in C^2 and could correspond to physical symmetries giving rise to different actions. $SL(2,Z)$ has 3 matrix generators $S = [0,1;-1,0]$,$T = [1,1;0,1]$ and $T^{-1} = [1,-1;0,1]$ acting linearly on (z^1, z^2). The product $U = ST$ is called $U = ST$ duality. S corresponds to inversion $z \to z$ and to electric magnetic duality. In M-theory there are 3 analogous dualities S, T, and U. One might guess that SL(2,Z) appears because it generates global conformal transformations of the conformal algebras.

2. In the TGD framework, the self-duality of CP_2 Kähler form implies that the electric and magnetic fluxes over a given 2-surface of CP_2 are indeed identical. This inspires the proposal that TGD is self-dual with respect to S [K1] a stronger condition would be that the entire $SL(2,Z)$ defines equivalent actions. In fact, in the TGD framework space-time surfaces are minimal surfaces apart from singularities irrespective of the action principle and the action makes itself manifest only through the singularities at which the minimal surface property fails [L23, L24, L25, L18].

   A weak form of this duality is proposed to be realized at the light-like interactions between Euclidean CP_2 type extremals and Minkowskian regions of the space-time surface. The corresponding parts in the action correspond to Kähler action and the associated topological instanton term.

3. In TGD, the geometry of the "world of classical worlds" (WCW) allows two kinds of symmetries.

   (a) The generalized super-conformal symmetries defined as a 4-D generalization of Kac-Moody symmetries appear since holography = holomorphy vision implies that space-time surfaces are holomorphic minimal surfaces of $H = M^4 \times CP_2$ irrespective of the action as long it is general coordinate invariant and constructed in terms of the induced geometry. One can assign the charges of these symmetries to the light-like boundaries of the causal diamond $CD = cd \times CP_2$ and also to the light-cone proper time constant surfaces of the light-cone.

   (b) Super-symplectic symmetries in turn have a natural action at the boundaries at the light-like orbits of the partonic 2-surfaces between the Euclidean interior of the CP_2 type extremals and the Minkowskian exterior. Could a strong form of holography, which would state that it is enough to define the theory using only the charges of either kind, make sense. Also this kind of duality might be seen as a kind of LC .

In the following I will discuss number theoretic and geometric LC in the way as I see them in the TGD framework. My view is not of a professional mathematician but of physicist and TGD dictates my views to a high degree. I have discussed LC several times already earlier [K3, K4] [L13, L23].

3.2 Number theoretic LC and TGD

In TGD both number theoretic and geometric LC are expected to play a role. Number theoretic LC is naturally associated with the adelic physics [L1, L2] involving number theoretic discretizations of the space-time surface in some extension of rationals. All p-adic number fields are involved.

This requires theoretical universality, which is guaranteed if the definition of space-time surfaces is such that it makes sense in all number fields. In holography = holomorphy vision, the definition of the space-time sheet as a root for a pair of polynomials, with coefficients in some extension E of rationals, guarantees this. In number theoretic vision one must have discretization by assuming that the roots are in rationals or even integers of E. One obtains a hierarchy of adeles labelled by E, forming an evolutionary hierarchy.

In the TGD framework, also a modification of the notion of adele can be considered [L20, L23]. p-Adic number fields and their extensions can be glued together along integers, which can be regarded as simultaneously belonging to several p-adic number fields since the p-adic integers n in the intersection of the p-adic number fields can be expressed as a power series of an integer m divisible by the primes considered. Systems characterized by different p-adic primes, identifiable as ramified primes for a polynomial, must interact, and the modified view of adeles allows us to describe this interaction at the level of discretization.

3.2.1 How could the number theoretic LC relate to TGD?

Some background is required in order to understand how the number theoretic Langlands involving finite fields and modular forms in Langlands group relate to TGD.

1. Number theoretical universality and holography = holomorphy principle

Number theoretical universality and holography = holomorphy principle [L23, L22] are central in TGD.

1. In TGD number theoretical universality becomes a basic physical principle [L23]. The number theoretical view and geometric view of physics are proposed to be equivalent. This M^8-H correspondence can be seen as a generalization of momentum-position duality of quantum mechanics required by the replacement of point-like particle with 3-surface and is analogous to the LC .

   An intuitive motivation for this proposal is that in free field theories, the field equations at the level of momentum space reduce to algebraic equations, typically mass shell conditions. At the level of space-time, one has partial differential equations describing the propagation of fields.

2. This view is realized in terms of holography = holomorphy correspondence implying that space-time surfaces are analogs of Bohr orbits [L23, L22]. The field equations characterizing the space-time surfaces reduce to equations, which are a non-linear generalization of massless field equations.

   There is a completely general solution ansatz for which the field equations reduce to minimal surface equations for any general coordinate invariant action principle constructed in terms of the induced geometry. At the level of H, the field equations reduce to purely algebraic equations involving a contraction of two tensor quantities, which are of different types as complex tensors and therefore vanish identically [L23]. Minimal surface equations fail only at the singularities, which have a dimension lower than 4 [L25, L18]. At the level of M^8, identifiable as octonions with number theoretical inner product $Re(o_1o_2)$, associative holography allows to construct the 4-surfaces in terms of 3-D holographic data [L21]. M^8-H duality allows an alternative manner to construct these 4-surfaces.

3. M^4 metric has Minkowskian signature and for the space-time surface the regions outside elementary particles (partonic orbits) are Minkowskian. For Minkowski signature one must generalize the notion of complex structure to Hamilton-Jacobi structure, as I call it [L19]. Hamilton-Jacobi structure is a fusion of ordinary complex structure and hypercomplex structure. Hypercomplex numbers are very much like complex numbers but do not form a number field since the norm vanishes for light-like hypercomplex numbers. Hypercomplex coordinate u as counterpart of complex coordinate z corresponds to a coordinate for which coordinate lines are light-like. Its conjugate v corresponds to a dual light-like coordinate.

4. In the general case, the space-time surfaces can be identified as roots of 2 analytic functions f_1 and f_2 of the 4 generalized complex coordinates of $H = M^4 \times CP_2$. One of them is a hypercomplex coordinate of M^4 and the remaining 3 are ordinary complex coordinates. The coefficients of the analytic functions can be also assumed to belong to an algebraic extension E of rationals and this gives rise to a hierarchy consisting of discrete subspaces of the "world of classical worlds" (WCW).

5. One can also consider a hierarchy of polynomial pairs (P_1, P_2) with coefficients in E. In the simplest case the coefficients are ordinary rationals or equivalently integers. One obtains a hierarchy of solutions of field equations labelled by the extension and by the degrees of the

polynomials and by the extensions of rationals that they define. This hierarchy is identified as an evolutionary hierarchy [L1, L2]. The higher the complexity of the space-time surfaces, the higher the evolutionary level of the system.

2. Number theoretic discretization gives cognitive representations

In the number theoretic vision [L23], the number theoretic discretization is a second aspect of TGD. It applies to space-time surfaces and, at a more abstract level, to the WCW coordinates defining moduli characterizing the space-time surfaces.

1. One can say that the most concrete cognitive representations correspond to number theoretic discretizations of the space-time surface. The first question concerns the number of roots $(P_1, P_2) = (0, 0)$ for a given prime p characterizing a p-adic number field, in particular the ramified prime of a related polynomial. Can one estimate the number of roots for any finite field in turn defining the number of solutions in corresponding p-adic number field? This is the key question.

2. From the example discussed in Frenkel's interview, I learned how one can get a rough idea of this number at least in some cases. This argument generalizes the well-known counting argument for the number of roots of two polynomials of two real variables. In this case the number of solutions is the number of variables minus the number of conditions. For polynomials defined in a finite field, the number of points (x, y) in a finite field is p^2. The Diophantine equation gives p conditions. Therefore the rough guess for the number of solutions is $p^2/p = p$ (in the discrete situation, subtraction is replaced by division). Already this is a fantastic piece of information.

3. Could it be possible to get the exact number # of integer solutions modulo p for any p-adic prime? LC in the case of the Diophantine equations considered in the interview comes to rescue. The modular function assignable to the equations codes in terms of its Taylor coefficients the deviations of the $\# - p$ from the rough guess for all primes p except a finite number of exceptional ones! Note however that the Diophantine equation possesses a modular symmetry.

3. Could the number theoretical LC generalize from dimension 2 to dimension 4?

Could the number theoretical LC for polynomials of two arguments generalize from dimension 2 to dimension 4. Holography = holomorphy vision suggests a very concrete formulation of this generalization.

Consider first the statement of the problem in the TGD context.

1. Cognitive representation as a discretization of the space-time surface would consist of the points of the space-time with coordinates in the extension E of rationals defining the coefficient field of P_1 and P_2. These points would be rational numbers as ratios of E integers. One should be able to count the numbers of the solutions to the equations $(P_1, P_2) = (0, 0)$ for which the $H = M^4 \times CP_2$ coordinates have values in E. A weaker condition is that only M^4 coordinates belong to E. Could a generalization of LC make this possible?

2. The 2-dimensional elliptic surfaces (analogs of string world sheets) reside in C^2, which has 2-complex dimensions. Space-time surfaces have 1 hypercomplex dimension and 1 complex dimension and have a generalized complex structure, Hamilton-Jacobi structure [L19]. Space-time surface has 2 generalized complex coordinates (u, w). u is a hypercomplex coordinate with light-like coordinate curves and w is a complex coordinate.

   The 2-complex dimensional space C^2 with complex coordinates (x, y) is replaced with the embedding space $H = M^4 \times CP_2$ having 4 generalized complex coordinates (one of them is hypercomplex) so that the polynomials P_1 and P_2 depend on 4 generalized complex coordinates (u, z_1, z_2, z_3).

3. An elliptic surface is a root of a single polynomial of degree 3 depending on (say) 2 complex variables (z_1, z_2) in 2-complex dimensional space C^2. In TGD, space-time surfaces are defined as roots of two polynomial equations $(P_1, P_2) = (0, 0)$.

4. The natural generalization of the 2-dimensional (in real sense) hyperbolic space H^2 (remember Escher) is the hyperbolic 3-space H^3. This corresponds to the mass shell as momentum space or light-cone proper time $a = constant$ surface in Minkowski space M^4 (actually inside a causal diamond cd). What is remarkable is that the Lorentz group $SL(2,C)$ and its modular subgroups act in H^3 and have the same interpretation as in special relativity.

Could one find a modular form in the product $H^3 \times CP_2$, where H^3 us hyperbolic 3-space coding for the numbers of the solutions to the number theoretical discretization conditions for the space-time surface for any pair $(P_1, P_2)$of polynomials? If this were possible, it would have enormous practical value. This might be possible except under very special conditions probably related to the ramified primes of the polynomials involved.

1. The counterparts of the holographic boundary conditions at selected hyperboloids $a = a_k$ should satisfy modular symmetries. Is this implied by the Bohr orbit property for the space-time surface [L23]? This would mean that the space-time surfaces are analogous to H^3 counterparts of discrete plane waves.

2. The rough estimate for the number of solutions of the cubic equation with modular invariance generalizes. In number theoretic discretization, the extension of rationals corresponds to a finite field $G(p,k)$ for some k. The number of elements in the finite field $G(p,k)$ is p^k. One has a p-adic discretization of the space-time surface using points for which the preferred coordinates are algebraic integers in an extension of p-adic numbers induced by the extension of rationals. These points are approximated with points of finite field $G(p,k)$. The generalized complex dimension of H is $D = 4$. Therefore one has p^{4k} points in the discretization. The two vanishing conditions $(P_1, P_2) = (0,0)$ give p^{2k} conditions so that the rough guess for the number of the solutions is p^{2k}. Already this is a fantastic result, which was new to me.

3. One might hope that if modular invariance is realized at the hyperboloids $H^3(k)$ defining the boundary data, then a modular form assignable to the pair (P_1, P_2) could give the number of roots in $G(p,k)$ by coding the correction $\# - p^{2k}$ by some coefficients in its Taylor expansion. This would in turn give the numbers of the p-adic roots since they can be generated once the solution in finite field $G(p,k)$ is known.

3.2.2 How to identify the Galois group in the number theoretic picture?

The physical interpretation would be that the Galois group extends the group algebra of ${}^LG$, where G is $SL(2,C)$ acting at H^3. The Galois group brings in additional degrees of freedom: a kind of Galois spin. There are two interpretations of the Galois group.

1. The simplest option is that K corresponds to an extension E of rationals appearing as the coefficient field of the polynomials or even as the field for the Taylor coefficients of analytic functions considered so that the number theoretic Galois group is naturally associated with it.

   In this case one cannot however identify the ramified primes since they require a definition of the extension in terms of the roots of a polynomial. The Galois group would not characterize the polynomial P_1 or P_2 but the field extension E. In this case, one would consider the discretization allowing only E-rational points.

   The group G could be the Lorentz group $SL(2,C)$ represented in the space of modular forms defined in H^3. The complex dual group ${}^LG$ would have the same Lie-algebra as $SL(2,C)$ and would be a semidirect product of ${}^LG^0 \ltimes Gal(E/Q)$.

2. In the TGD framework, one can also consider the identification of the Galois group as that for the extension of E defined by the space-time surface itself, if in the case that it is well-defined. If holography = holomorphy vision is accepted, H has 4 generalized complex coordinates. Suppose that 2 of the complex coordinates are E rationals. One can solve the 2 remaining generalized complex coordinates from the conditions $(P_1, P_2) = (0,0)$.

   The hypercomplex coordinate of M^4, which has light-like coordinate curves and is real and therefore does not allow complex values, is not a good candidate to be one of these 2 coordinates. This leaves only two options under consideration. The two complex coordinates

ξ^1, ξ^2 of CP_2 would make sense when the space-time surface has 4-D CP_2 projection and the induced metric is Euclidean.

The second option corresponds to the complex coordinate ξ^1 or ξ^2 of CP_2 and to the complex coordinate w of M^4 assignable to its Hamilton-Jacobi structure [L19]. The values of, say, ξ^1 and w are obtained by solving $P_1 = 0$ and $P_2 = 0$. One could require that w is E rational and ξ^1 in the extension of E or vice versa. One can assign to both options a Galois group and an extension of E.

3. This picture would conform with the fact that in TGD two kinds of effective Planck constants appear naturally. The first kind of effective Planck constant, identified tentatively as the dimension of extension of E, can be assigned with the space-time surface as a covering of M^4 and would be relatively small since one does not expect that the space-time has very many sheets with the same M^4 projection. The other effective Plank constant corresponds to the space-time surface as covering CP_2 and could be the counterpart for say periodic field pattern having as a TGD counterpart a bundle of monopole flux tubes in H. The number of the flux tubes would correspond to the degree of the polynomials considered and could be very large [L3, L10, L11, L17].

3.2.3 Objections

The best way to sharpen a hypothesis is to invent objections against it.

1. According to Frenkel, there are exceptional primes for which the coding fails. A good guess is that they correspond to the ramified primes for a cubic polynomial in the case considered. Indeed if the other coordinate of the cubic polynomials is integer, the polynomial reduces to a cubic polynomial with integer coefficients and one can assign to it roots and ramified primes.

   In TGD, the p-adic primes characterizing space-time surfaces are identified as the ramified primes assignable to the polynomials defining the space-time surface. If the problematic primes are ramified primes, the coding would fail for the physically interesting primes! The blessing in disguise is that the number of ramified primes is finite.

2. Modular invariance characterizes the example considered by Frenkel. Modular group acting as symmetries is analogous to lattice translations: torus topology indeed reflects this. Modular invariance implies that the numbers of solutions to a given cubic equation are coded by a particular modular invariant function. Could the TGD analog of this condition, stating that the solutions are analogous to plane waves, be too strong? One might of course hope that by the notion of holography meaning that space-time surfaces are analogs of Bohr orbits requires this condition or that it corresponds to "good" representations of $SL(2,C)$ at quantum level.

Suppose that a modular form for, say 3-D hyperbolic space H^3 (mass shell $a = a_k$), invariant under a modular group identified as discrete subgroup of $SL(2,Z)$ (or $SL(2,Z_E)$), codes for the numbers of say solutions to $(P_1, P_2) = (0,0)$ belonging to an extension E of rationals appearing as coefficients of P_i. This would mean that all 4 generalized complex coordinates belong to E.

1. One might expect that the modular group acts as symmetries of the space-time surface in the same way as the modular group for elliptic surfaces. This could mean periodicity analogous to that for planewave solutions to field equations. Could the TGD counterparts of plane wave solutions of free field theories correspond to generalized elliptic functions of H^3 for which 4-D translational invariance is replaced with 3-D modular invariance at H^3. Inside the space-time surface translational (or scaling-) invariance with respect to a is probably too much to require but holography = holomorphy principle allows the continuation of the boundary data to the space-time interior so that it is not needed. The modular invariance would reduce to the modular invariance of the holographic data at some surfaces H^3 with constant value of the light-cone proper time.

   One would obtain the H^3 analogs of 3-D plane waves in E^3. In free field theory in M^4, E^3 plane waves can be continued to 4-D plane waves. In TGD, holography allows to continue the counterparts of the E^3 plane waves in H^3 to the interior of the space-time surface.

Could it be that modular invariance is a physically sensical restriction.

1. One cannot have a linear superposition of the H^3 plane waves since TGD is not a free field theory. This restriction holds true for the so-called massless extremals [K1] for which the linear superposition holds only for modes propagating in the same direction: one has an analog of a laser beam. The pulse shape is preserved in the propagation since there is no dispersion.

2. Linear superposition would correspond in TGD to a union of space-time surfaces having the same M^4 projection: a test particle touching the space-time surfaces in the union would experience the sum of the induced gauge fields and gravitational field associated with different space-time sheets.

3. How strong conditions does the modular symmetry pose on the polynomials (P_1, P_2)? Does modularity mean that only a very restricted number of polynomials allows this coding? Or could all solutions of field equations reduce to H^3 analogs of plane waves for some set of $a = constant$ hyperboloids H^3?

   This form of holography I have indeed proposed. This would mean a nonlinear generalization of Fourier analysis from $E^3 \subset M^4$ to the level of the space-time surface with E^3 replaced with the intersections of these special subset of hyperboloids. The intersections of the space-time sheet with $H^3(a_k)$ would represent the Fourier mode and their superposition would correspond to a set theoretic union.

4. I have proposed that the tessellations of H^3 are fundamental in TGD and characterize the solutions of the field equations as symmetries of the boundary data at a finite set of hyperbolic 3-surfaces identifiable as light-cone proper time $a = a_k$ surface. The equations for the minimal surfaces are not completely deterministic and non-determinism would be localized at these hyperboloids. At these hyperboloids the data would satisfy the modular symmetry so that the intersection with $H^3(a_k)$ would reduce to a union of fundamental domains of the modular subgroup, which are analogous to lattice cells in E^3.

   Are space-time surfaces without modular symmetry for any $H^3(a)$ impossible or are they such that the representations of $SL(2, C)$ are not "good"? In the case of H^2 this poses extremely powerful conditions on the polynomials considedere and the same is expected in the case of H^3.

3.3 Geometric LC and TGD

Also geometric LC would emerge with the holography = holomorphy vision [L23, L24, L22]. In this case, number fields are replaced with function fields defined by generalized holomorphic functions in $H = M^4 \times CP_2$. The pairs (P_1, P_2) of polynomials with coefficient field E as extension of rationals define space-time surfaces as their roots. These polynomial pairs form a function field so that space-time surfaces as roots $(P_1, P_2) = (0, 0)$ behave like numbers and form an analog of a number field. Note that here also rational functions can be allowed since their poles do not appear as points of the space-time surface.

Now the notion of the Galois group must be generalized. The generalized Galois group must permute different regions of the space-time surface as roots for the pair $P_1, P_2)$ of polynomials and generalize holomorphic maps of H to itself should define the transformations performing these permutations.

3.3.1 Space-time surfaces as numbers

One can endow the polynomial pairs defining the space-time surfaces with a structure of a function field, which is inherited by the roots so that also space-time surfaces can be regarded as numbers.

1. One can define basic arithmetic operations (multiplication, division, sum and subtraction) for the space-time surfaces defined by the conditions $(P_1, P_2) = (0, 0)$ and $(Q_1, Q_2) = (0, 0)$. The product of space-time surfaces would be defined by the conditions $(P_1Q_1, P_2Q_2) = (0, 0)$. The generalization to the other arithmetic operations is obvious. For the componentwise product of two polynomial pairs, one obtains all four root pairs as roots.

2. One can multiply the defining polynomials by functions which do not vanish in the region defined by the space-time surface. This region corresponds naturally to the causal diamond *cd* of M^4. In this case the space-time surface as a root is not affected. This is analogous to the multiplication of a polynomial with a polynomial (or even rational function), which vanishes nowhere in the range considered. This multiplication corresponds to a multiplication of an ordinary polynomial with a rational number.

3. One can also construct functional composites of (Q_1, Q_2) and (P_1, P_2) as $(Q_1 \circ P_1, Q_2 \circ Q_2)$. If one has $Q_i(0) = 0$, the roots $(P_1, P_2) = (0, 0)$ are not affected but one obtains additional roots as non-vanishing roots of (Q_1, Q_2). If Q_i have no non-zero roots in the region considered, the process acts as a symmetry transformation. The action on function pairs (P_1, P_2) having no roots in the region considered would be trivial.

How does this picture relate to the notion of infinite-primes [K6]? Infinite primes can be said to correspond to a hierarchy of second quantizations of a supersymmetric arithmetic theory in which single boson- and fermion states at a given level correspond to infinite primes as many-particle states of the previous level. At the lowest level they correspond to ordinary primes having a possible interpretation as momenta.

1. At a given level Dirac vacuum X corresponds to the product of all primes, which is infinite as a real number but finite as a p-adic number for any prime p.

2. The construction of infinite primes as analogs of free particles is analogous to the construction of many particle states in second quantization. The simplest states are of the form $P = mX + n$, where m and n are integers chosen in such a way that P is indivisible by any finite prime.

3. Also infinite primes, analogous to bound states, are possible and are realized as irreducible polynomials $P(X)$ (no decomposition to a product of monomials with integer coefficients) with coefficients, which are (possibly infinite) primes of the previous level. At level n, these polynomials can be regarded as polynomials of n formal variables $X_1, ...X_n$.

4. At the lowest non-trivial level the roots of P correspond to integers and possibly complex algebraic numbers. At the next level, one has polynomials of two variables and roots correspond to 1-D curves. Next level gives 2-D surfaces, third level gives 3-D surfaces and the fourth level 4-D surfaces as roots. This construction generalizes from the case of rational integers to integers in extensions of rationals.

5. The proposal has been that this hierarchy corresponds to the scale hierarchy of space-time sheets with sheets having particle interpretation. The challenge is to see whether this hierarchy could somehow relate to the hierarchy defined by the pairs (P_1, P_2) of polynomials.

   $P_1 = 0$ and $P_2 = 0$ surfaces define the analogs of 6-D twistor spaces of M^4 and CP_2 having the same space-time surface X^4 as a base space and twistor sphere as a fiber [L23, L24, L22]. The intersection of these two twistor spaces gives X^4. H has 4 generalized complex coordinates so that a 4-level hierarchy for which X can be complex is suggestive. Why would the higher levels of the hierarchy be absent?

6. The geometric interpretation of the infinite hierarchy of infinite primes encourages the question whether one could obtain also the higher levels of the hierarchy in the holography= holomorphy vision in some way?

   It seems that one must consider H^n instead of H at the n^{th} level of the hierarchy. At the level of the spinor structure this means n^{th} tensor power of the spinor space of H. This conforms with the idea of repeated second quantization and with the ideat the infinite primes define an abstraction hierarchy of statements about statements about.... Note also that in condensed matter physics, the n-particle system is defined in terms of particles in $(E^3)^n$. This could be replaced by H^n, where n is analogous to the number of particles and to the level of the hierarchy.

   Could 4-D space-time surface in H^n correspond to the roots of polynomial-tuples $(P_1, P_2, ...P_k)$ of $k = 4n - 4$ polynomials depending on the $4n$ generalized complex coordinates of H^n? The roots satisfy $k = 4n - 4$ conditions eliminating this number of degrees of freedom so that the conditions define a 2-complex-dimensional surface of H^n identifiable as the space-time surface. Its projections to the n Cartesian factors of H^n would define 4-D surfaces, which could represent the geometric view of a particular particle-like entity about the entire state.

Could this description give a purely geometric description for the interactions of the particles? Or could it give a description for the hierarchy of space-time sheets topologically condensed at larger space-time sheets. The mathematical description of this hierarchy is still far from quantitative. This requires ordering. In the case of infinite primes there is a natural ordering involved. Does this generalize? At n^{th} level one has polynomials of n:th variable having as its coefficient polynomials of $(n-1)^{th}$ level. The ordering is partially determined by requiring that a given level has the highest possible polynomial degree. At each level this selects a finite number of alternatives for the coordinate in question.

3.4 How to define the geometric counterpart for rationals and their extensions geometrically?

How do the notions of rationals and extension of rationals generalize? Does the generalized Galois group leave the generalized rationals invariant? A natural approach is provided by zero energy ontology (ZEO) in which the causal diamonds $CD = cd \times CP_2$ play a key role as analogs of perceptive fields of a conscious entity, the observer.

Polynomial pairs (Q_1, Q_2), which are non-vanishing inside CD would define the counterpart for rationals since the element-wise multiplication of (P_1, P_2) to give (Q_1P_1, Q_2P_2) does not give rise to new roots inside CD. Note that also rational functions can be considered. These counterparts of rationals have no roots so that the generalized Galois group permuting the space-time regions representing various roots acts on them trivially.

Extensions of these generalized rationals correspond to the space-time surfaces defined by polynomial pairs (P_1, P_2) having roots inside CD. Different roots correspond to space-time regions and they define the extension. Galois groups realized as generalized homomorphisms of H must map these regions to each other.

This proposal raises interesting questions. What are the counterparts of the roots of polynomials as space-time surfaces? What is the counterpart of discriminant, which allows to assign to a polynomial a seet of ramified primes as rational primes. What is the space-time counterpart of rational prime and its

1. What are the counterparts of the roots of polynomials as space-time surfaces? For polynomials that the polynomial factorizes to a product of monomials $x - r_n$ with roots which are the roots of polynomials. In the recent case the corresponding decomposition in the case of $P_1 = 0$ given by a polynomial having as its only root the space-region correspond to a given root of P_1 having its coefficients in the base field but with the root as algebraic number with the solution of some of the 4 complex variables appearing in P_1 and solved in terms of others as an algebraic function with coefficients depending on the point of the region defined. One would have $z_k - r_k(\{z_l, l \neq k\}$, where the algebraic function r_k gives z_k locally as a solution of $P_1 = 0$.

   Root as an algebraic number is replaced with algebraic function and defines the root as a space-time region. One can say that the surface decomposes to a union of regions defined by the roots of monomials, which join along their boundaries at which two roots co-incide.

2. There are preferred choices for the choice of z_k. The natural condition is that the degree of P_1 as a polynomial of z_k is maximal. The cusp catastrophe $dV(x)/dx = x^3 + 2ax + b = 0$ illustrates the situation: behavior variable x is preferred choice for the counterpart of z_k: the roots for the parameters (a, b) are rational functions because a and b appear linearly.

   Now one has a pair of polynomials and the decomposition can be carried out for both P_1 and P_2. The roots for the pair are intersections of the roots of P_1 and P_2. One can substitute to P_2 $z_{k_1} = r_k(\{z_l, l \neq k\}$ and solve some coordinate z_{k_2}, $k_1 \neq k_2$ as an algebraic function of the other remaining parameter like coordinates.

Do the notions of discriminant and ramified prime as rational prime generalize?

1. Consider also now $P_1 = 0$ as a simpler example. One can decompose P_1 to monomials which are algebraic functions $z_k = r_k$. Since these functions are elements in an extensions of the function field of rational polynomials, one can also form the differences $z_i - z_j$ and define the discriminant as the local product $\prod_{i \neq j}(z_i - z_j)$. For the ordinary polynomials the discriminant decomposes to primes of the coefficient field, say rationals.

   In this case the discriminant should decompose to analogs of primes as rational polynomials. Irreducible rational polynomials of prime degree p are primes in the sense that theyt cannot be decomposed into polynomials of lower degree without leaving the coefficient field. There the ramified primes could correspond to the factors of the discriminant with prime degree.

2. An interesting question is how the prime degrees appearing of the polynomial prime factors of P_1 relate to the ramified primes assignable to the roots of P_1 when restricted to physical special 2-surfaces X^2 fixed to some degree by the condition that $P_{1|X^2}$ has coefficients in some base field E as extension of rational for a suitable chosen complex coordinate for X^2 most naturally one of the generalized complex coordinates used. $P_1 = 0$ and $P_2 = 0$ give 2 conditions giving the space-time surface X^4.

3. An interesting question relates to general coordinate invariance. In particular, to the possibility to perform holomorphic coordinate changes. Physically the roots of polynomials whose arguments are replaced with functions of new coordinates, does not affect the roots geometrically so that this is not a problem. In some preferred coordinates one obtains polynomials and the physical intuition serves as a guideline in the identification of these preferred complex coordinates. The light-like hypercomplex coordinate and complex coordinate w associated with the Hamilton-Jacobi structure of M^4 and complex Eguchi-Hanson coordinates of CP_2 are natural candidates for the preferred coordinates.

Ordinary rational primes are of special physical interest in TGD. How could these rational primes, or more generally, E-rational primes emerge?

1. Is the polynomial pair (P_1, P_2) enough or are two additional polynomials (P_3, P_4) needed so that the one would obtain a hierarchy consisting of 6-surfaces as roots of P_i having interpretation as analogs of twistor space of the space-time surface X^4 as their interaction, 2-surface X^2 and a discrete set of points. Since one coordinate is a light-like hypercomplex coordinate. Can one select (P_3, P_4) freely or are they determined by the pair (P_1, P_2)? This would be the case if the surfaces X^2 and the roots of a polynomial at it are determined by some geometric and physical conditions.

2. The 2-surfaces X^2 might actually correspond to their metrically 2-D light-like orbits X^3. Could the partonic orbit correspond to intersection of two roots of (P_1, P_2). as regions X^2 (or X^3). In the case of partonic orbits X^3, these roots could correspond to Euclidean and Minkowskian space-time regions and at their interface the induced metric would be degenerate and effectively 2-D. Therefore the partonic orbits are represented as roots $(P_1, P_2, P_3) = (0, 0, 0)$. Note that the coefficients of P_3 depend on the root of $(P_1, P_2) = (0, 0)$ so that each root corresponds to a different P_3.

3. The fact that partonic orbits are determined by the condition $(P_1, P_2) = 0$ alone, suggests that P^3 cannot be chosen freely in this case. On the other hand, the light-likeness of the partonic orbits might mean non-determinism (for CP_2 type extremals as vacuum extremals of Kähler action, M^4 projection is an arbitrary light-like curve).

   The conditions $P_1 = 0$ and $P_2 = 0$ give z_{k_1} and z_{k_2} as algebraic functions of 2 generalized complex coordinates (w, ξ) at the space-time surface. When one substitutes z_{k_1} and z_{k_2} as to the third polynomial $P_3(z_{k_1}, z_{k_2}, w, \xi)$ as a function of (w, ξ), one should obtain a polynomial of (w, ξ) with coefficients in E. This is a highly non-trivial condition. The roots of P_3 should define partonic orbits X^3.

One wants to assign p-adic primes as ramified primes to the ordinary roots of P_4 restricted to a given X^3 and having form depending on it so that each partonic orbit could correspond to its own p-adic prime as some ramified prime. Algebraic approach is the first option to be considered.

1. The roots should correspond to the roots of a polynomial P_4 of 4 generalized complex coordinates subject to the condition $(P_1, P_2, P_3) = (0, 0, 0)$ defining X^2 or X^3 so that one would have $(P_1, P_2, P_3, P_4) = (0, 0, 0, 0)$.

   The roots could correspond geometrically to light-like curves at metrically 2-D partonic orbits, that is, fermion lines to which one wants to assign ramified primes as p-adic primes. Also now one can ask whether it is possible to P_4 freely as polynomials having coefficients in E does (P_1, P_2) pair fix P_4 completely? Can one identify a physical condition fixing the light-like partonic orbits of fermions? Fermion lines should correspond to the intersections of string world sheets as 2-D singularities of X^4 with X^3. Could this condition fix P_4 so that it is determined by the pair (P_1, P_2).

2. If P_4 is a polynomial of 4 generalized complex coordinates with coefficients in E, it should remain such at the partonic orbit. If 3 generalized complex coordinates are E-rational constants at the partonic orbit, then $P_4(z)$ has coefficients in E and and its roots are in an extension of E and one can assign ramified primes to the roots of P_4. E-rational constancy conditions are very powerful. Note that analogous conditions are encountered for the elliptic curves: in this case the rational points form a lattice due to the underlying symmetry of the elliptic curves. This looks like a rather complex approach. Something more elegant is needed.

The purely algebraic approach looks clumsy. Could a geometric approach based on the physical picture be more elegant?

1. The conditions defining light-like partonic orbits ($\sqrt{g_4} = 0$) and the conditions defining light-like fermion lines as intersections of string world sheets as singularities with partonic orbits could be the geometric conditions. This suggests that one should find a manifestly general coordinate invariant formulation, where the polynomial form emerges only in preferred coordinates and ramified primes emerge naturally.

2. Concerning the identification of ramified primes, one could also start from the situation in which the n roots at X^2 are assumed to correspond to the intersections of string world sheets and partonic orbits. One can introduce a complex coordinate z. The points should be algebraic points for this choice of coordinate and therefore expressible as roots of a polynomial P of degree n with coefficients in E. This condition gives very powerful conditions on the coordinate z. If some choice of z is found, one can write the polynomial as a product of monomials defined by the roots and can calculate the discriminant and the ramified primes associated with it.

   How unique is this complex coordinate? The root differences are only scaled under linear modular transformations so that the spectrum of ramified primes is preserved. If these linear transformations define the allowed global holomorphies, the preferred complex coordinate at X^2 is rather unique.

3.4.1 The counterpart of the Galois group in the geometric LC

Recently I considered a view of the Galois group, which conforms with the geometric Langlands in which the roots of a polynomial are replaced with space-time regions as roots of the pair (P_1, P_2).

1. The ordinary Galois group is assigned with the roots of P defining an extension of E. The Galois group acts as a permutation group of the roots of a polynomial of a polynomial P with integer coefficients. Besides this it acts trivially in k if K is an extension of k.

2. For a pair (P_1, P_2) the roots $(P_1, P_2) = (0, 0)$ correspond to regions of a connected 4-surface. One should generalize the notion of Galois group so that it permutes various roots of $(P_1, P_2) = (0, 0)$ as regions of the 4-D surface.

   Could generalized holomorphic transformation represent the action of the generalized Galois group on the space-time surface as flows permuting the regions representing the roots [L23]. This transformation would not only map the roots as space-time regions to each other but also respect the local root property. This might pose restrictions to the Galois group in the sense that the full permutation group would not be allowed.

3. If this flow reduces to isometries, then the action must reduced to that of a discrete subgroup of $SO(3)$ or $SL(2,C)$ and one obtains that the allowed Galois groups correspond to the hierarchy of discrete subgroups of $SO(3)$ associated with inclusions of hyper-finite factors [L23] and with McKay correspondence [A10, A9, A7, A6, A5] [L4].

4. What about the analogy for the condition that the Galois group leaves the field E invariant? The natural identification is that the counterparts of the field E are pairs of polynomials (or rational functions, or even analytic functions), which are non-vanishing inside CD. The notion of root indeed generalizes also to analytic functions so that the notion of the geometric Galois group is number-theoretically universal.

3.4.2 The identification of the geometric Langlands group

Just as in the case of the number theoretical LC, the geometric Langlands group would correspond to the semidirect product of ${}^{L}SL(2,C)^{o}$ with a Galois group, which would be now the geometric variant of the Galois group. $SL(2,C)$ and its subgroup $SL(2,Z_E)$ would act on the selected discrete set hyperboloids $H^3(a_k)$.

An additional hypothesis, giving hopes for obtaining the numbers of the numbers of p-adic roots $(P_1,P_2)=(0,0)$, is that the Bohr orbitology forces a modular invariance in the sense that the boundary data of holography are analogous to plane waves with a definite discretized 3-momentum in the sense that a discrete subgroup of $SL(2,Z_E)$ defines a periodic tessellations of the H^3 projection of the space-time surface defining boundary data of the holography. The plane waves would correspond to modular forms in the hyperboloid H^3 covariant under $SL(2,Z_E)$.

Also in the CP_2 degrees of freedom analog of modular invariance might hold true for a discrete subgroup of CP_2 so that the 3-surface in CP_2 degrees of freedom would be an analog of Platonic solid. This would conform with the quantum classical correspondence suggested by the Bohr orbitology and suggest that space-time surfaces reflect the quantum numbers of the fermionic quantum states associated with them.

4 Appendix

In the following some notions of algebraic geometry, group theory, and number theory are briefly explained.

4.1 Some notions of algebraic geometry and group theory

4.1.1 Notions related to modular forms and automorphic forms

Fuschian and modular groups are discrete subgroups of $SL(2,R)$ acting as invariance groups of modular functions.

1. Fuschian groups (`https://cutt.ly/hBnOYJU`) is a discrete subgroup of $PSL(2,R)$. The group $PSL(2,R)$ can be regarded equivalently as a group of isometries of the hyperbolic plane, or conformal transformations of the unit disc, or conformal transformations of the upper half plane, so a Fuchsian group can be regarded as a group acting on any of these spaces. There are some variations of the definition: sometimes the Fuchsian group is assumed to be finitely generated, sometimes it is allowed to be a subgroup of $PGL(2,R)$ (so that it contains orientation-reversing elements), and sometimes it is allowed to be a Kleinian group (a discrete subgroup of $PSL(2,C)$), which is conjugate to a subgroup of $PSL(2,R)$.

   Fuchsian groups are used to create Fuchsian models of Riemann surfaces. In this case, the group may be called the Fuchsian group of the surface. In some sense, Fuchsian groups do for non-Euclidean geometry what crystallographic groups do for Euclidean geometry. Some Escher graphics are based on them (for the disc model of hyperbolic geometry).

2. Modular group (`https://cutt.ly/hBgbH9S`) is the projective special linear group $PSL(2,Z)$ of 2×2 matrices with integer coefficients and determinant 1. The matrices A and A are identified. The modular group acts on the upper-half of the complex plane by fractional

linear transformations, and the name "modular group" comes from the relation to moduli spaces, such as the moduli space of conformal structures of torus.

Second presentation is transformations of the complex plane as Möbius transformations $z \rightarrow (az+b)/(cz+d)$ mapping upper plane and real axis to itself. $SL(2,R)/SL(2,Z)$ gives rise to a hyperbolic geometry identifiable as a fundamental domain of the tessellation of H^2 analogous to the lattice cell of the Euclidean planar lattice.

Modular group is generated by relations generators $z \rightarrow -1/z$ and $T: z \rightarrow z+1$. Modular group has a presentation $S^2 = I$, $ST^3 = I$. By posing the additional relation $T^n = 1$ one obtains a congruence subgroup denoted by $D(2,3,n)$.

These groups have generalization to discrete groups of $SL(n,C)$ and $Sl(n,R)$.

Modular forms and theta functions are closely related entities as also L-functions and generalize zeta functions.

1. A modular form (`https://cutt.ly/3BgbLsr`) is a (complex) analytic function on the upper half-plane satisfying a certain kind of functional equation with respect to the group action of the modular group, and also satisfying a growth condition. The theory of modular forms therefore belongs to complex analysis but the main importance of the theory has traditionally been in its connections with number theory. Modular forms appear in other areas, such as algebraic topology, sphere packing, and string theory.

   A modular function is a function that is invariant with respect to the modular group, but without the condition that $f(z)$ be holomorphic in the upper half-plane (among other requirements). Instead, modular functions are meromorphic (that is, they are holomorphic on the complement of a set of isolated points, which are poles of the function).

   Modular form theory is a special case of the more general theory of automorphic forms which are functions defined on Lie groups which transform nicely with respect to the action of certain discrete subgroups, generalizing the example of the modular group $\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{SL}_2(\mathbb{R})$.

   For instance, modular forms can be defined in a generalized upper half plane, which consists of symmetric $Gl(n,C)$ matrices such that the imaginary parts of the matrix elements are positive. For certain. values of n these spaces serve as moduli spaces for the conformal equivalence classes of Riemann surfaces and in the TGD framework elementary particle vacuum functionals as "wave functions" in WCW are identified as modular invariant modular forms in Teichmüller spaces [K2].

2. Theta functions (`https://cutt.ly/bBEFAe5`) are special functions of several complex variables. They are involved with Abelian varieties, moduli spaces, quadratic forms, and solitons. As Grassmann algebras, they appear in quantum field theory.

   For instance, the formula for Jacobi's theta function $\theta_1(z,q)$ reads as

$$\begin{aligned}\theta_1(z,q) &= 2q^{\frac{1}{4}}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\sin((2n+1)z) \\ &= \sum_{n=-\infty}^{\infty}(-1)^{n-\frac{1}{2}}q^{\left(n+\frac{1}{2}\right)^2}e^{(2n+1)iz} \quad .\end{aligned} \tag{4.1}$$

   The most common form of theta function is that occurring in the theory of elliptic functions. With respect to one of the complex variables (conventionally called z), a theta function has a property expressing its behavior with respect to the addition of a period of the associated elliptic functions, making it a quasiperiodic function. In the abstract theory this quasiperiodicity comes from the cohomology class of a line bundle on a complex torus, a condition of descent.

   One interpretation of theta functions when dealing with the heat equation is that "a theta function is a special function that describes the evolution of temperature on a segment domain subject to certain boundary conditions".

3. Dirichlet series correspond to L-functions and zeta functions. A Dirichlet series `https://cutt.ly/rBgbNKZ` is any series of the form $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$, where s is complex, and a_n is a complex sequence. It is a special case of the general Dirichlet series.

   Dirichlet series play a variety of important roles in analytic number theory. The most usually seen definition of the Riemann zeta function is a Dirichlet series, as are the Dirichlet L-functions.

Modular forms and L-functions correspond to each other.

1. Mapping of modular forms to L-functions assigns to the Fourier sum $\sum a_n q^n$, $q = exp(i2\pi z)$ of a modular form, also known as theta function (`https://cutt.ly/QBEYRfW`), an L-function defined as $\sum a_n n^{-s}$.

   Jacobi theta function $\theta(z) = \sum_{n=1}^{\infty} q^{n^2}$,$q = exp(i\pi z)$ has $\zeta(2s)$ as associated L-function.

2. Mellin transform of function f is defined as $M(f)(s) = \int_0^{\infty} dx x^{s-1} f(x)$ (`https://cutt.ly/gBEbWW4`). $\zeta(s)$ can be written as $(1/\Gamma(s))M(f(x))$, $f(x) = 1/(e^{-x}/(1-e^{-x}))$ identifiable as a partition function of harmonic oscillator with a energy spectrum consisting of positive integers.

4.1.2 Some group theoretic notions

Group theoretical notions.

1. Reductive groups

According to the Wikipedia article (`https://cutt.ly/9Bgbv9o`), a reductive group is a linear algebraic group over a field. One definition is that a connected linear algebraic group G over a perfect field (`https://cutt.ly/IBxHw9S`) is reductive if it has a representation with a finite kernel, which is a direct sum of irreducible representations.

Note that for any polynomial over a perfect field K all roots are in K, whereas for algebraically closed field they always have a root in K, as a matter of fact the number of roots equals to the degree of the polynomial in this case.

This does not say much to a layman. The fact that the every finite normal subgroup of a reductive group is central, is more informative. For instance, the Galois groups for extensions of extensions fail to satisfy this condition in general so that only simple Galois groups of Galois groups for which normal subgroups are central, are reductive.

Reductive groups include general linear group $GL(n)$ of invertible matrices, special linear group $SL(n)$ (in particular $SL(2,k)$), the special orthogonal group $SO(n)$, and the symplectic group $Sp(2n)$. Simple algebraic groups (in particular $SU(n)$) and (more generally) semisimple algebraic groups are reductive.

Claude Chevalley showed that the classification of reductive groups is the same over any algebraically closed field. In particular, the simple algebraic groups are classified by Dynkin diagrams, as in the theory of compact Lie groups or complex semisimple Lie algebras. Reductive groups over an arbitrary field are harder to classify, but for many fields such as the real numbers R or a number field, the classification is well understood. The classification of finite simple groups says that most finite simple groups arise as the group $G(k)$ of k-rational points of a simple algebraic group G over a finite field k, or as minor variants of that construction.

2. Borel subgroups, parabolic subgroups and parabolic induction

1. In the theory of algebraic groups, a Borel subgroup (`https://cutt.ly/jBgbmRX`) of an algebraic group G is a maximal Zariski closed and connected solvable algebraic subgroup. In Zariski topology the closed sets are algebraic surfaces, whereas in ordinary topology the set of closed sets is much larger. Zariski topology is therefore rougher than standard topology.

   For example, in the general linear group GL_n, the subgroup of invertible upper triangular matrices is a Borel subgroup. For groups realized over algebraically closed fields, all Borel subgroups are conjugate to this group.

2. Subgroups between a Borel subgroup B and the ambient group G are called parabolic subgroups. Parabolic subgroups P are characterized by the condition that G/P is a complete projective variety defined as by a vanishing conditions for a set homogeneous polynomials so that the solutions possess scale invariance. For algebraically closed fields, the Borel subgroups turn out to be the minimal parabolic subgroups in this sense. Thus B is a Borel subgroup when the homogeneous space G/B is a complete variety, which is "as large as possible".

3. According to the Wikipedia article (`https://cutt.ly/SBxTqTU`), parabolic induction is a method of constructing representations of a reductive group from representations of its parabolic subgroups.

   If G is a reductive algebraic group and $P = MAN$ is the Langlands decomposition of a parabolic subgroup $P \subset G$, then parabolic induction consists of taking a representation of MA, extending it to P by letting N act trivially, and inducing the result from P to G. Induction means extension of the represention of P to G. For instance, the representations of Poincare group can be induced from the representations of $SO(3) \times T^4$. That G/P is a complete projective variety must play an important role in this process.

3. Definition of L-group

According to Wikipedia, in representation theory the Langlands dual ^LG (`https://cutt.ly/cBgbTGs`) of a reductive algebraic group G (also called the L-group of G) is a group that controls the representation theory of G. If G is defined over a field k, then ^LG is an extension of the absolute Galois group of k by a complex Lie group. There is also a variation called the Weil form of the L-group, where the Galois group is replaced by a Weil group. The letter "L" in the name also indicates the connection with the theory of L-functions, particularly the automorphic L-functions. The Langlands dual was introduced by Langlands in a letter to A. Weil.

According to this definition ^LG would be a Lie group and contain the semidirect product of Galois group and of algebraic group over the extension of rationals. Note that amalgamated free product involves a third group U having embeddings to both Gal and $G(k)$ and $G(k)$ and Gal are "glued" along U.

4.1.3 Automorphic representations and automorphic functions

I am not a number theory professional, and in the following I can only try to demonstrate that I have at least done my best in trying to understand the essentials of the description of [A4] for the route from automorphic adelic representations of $GL_e(2,R)$ to automorphic functions defined in upper half-plane. A brief summary of the automorphic representations in Wikipedia involves the following key points.

1. One has an adelic analogy of group algebra, that is the space of functions in the adelic group G satisfying some additional conditions. Representation functions are left invariant with respect to the algebraic diagonal subgroup G_{diag}. Central character is interpreted as a map ω: $Z(K) \setminus Z(A)^\times \to C$.

2. Representation functions are finite sums of the left translates of function f by elements of adelic G. G acts from right on these functions. One speaks of a space of cusp forms with a central character ω.

3. A decomposition of the cuspidal representation into a direct sum of Hilbert spaces with finite multiplicities takes place.

The following describes the construction for $GL(2,Q)$, which is very relevant for TGD since $SL(2,C)$ acts as a covering of the Lorentz group.

1. Characterization of the representation

The representations of $GL_e(2,Q)$ are constructed in the space of smooth bounded functions $GL_e(2,Q)\backslash GL_e(2,A) \to C$ or equivalently in the space of $GL_e(2,Q)$ left-invariant functions in $GL_e(2,A)$. A denotes adeles and $GL_e(2,A)$ acts as right translations in this space. The argument generalizes to arbitrary number field F and its algebraic closure $\overline{F}$.

1. Automorphic representations are characterized by a choice of a compact subgroup K of $GL_e(2,A)$. The motivating idea is the central role of double coset decompositions $G = K_1AK_2$, where K_i are compact subgroups and A denotes the space of double cosets K_1gK_2 in the general representation theory. In the recent case the compact group $K_2 \equiv K$ is expressible as a product $K = \prod_p K_p \times O_2$.

   To my best non-professional understanding, $N = \prod p_k^{e_k}$ in the cuspidality condition gives rise to ramified primes implying that for these primes one cannot find $GL_2(Z_p)$ invariant vectors unlike for others. In this case one must replace this kind of vectors with those invariant under a subgroup of $GL_2(Z_p)$ consisting of matrices for which the component c satisfies $c \ mod \ p^{n_p} = 0$. Hence for each unramified prime p one has $K_p = GL_e(2, Z_p)$. For ramified primes K_p consists of $SL_e(2, Z_p)$ matrices with $c \in p^{n_p} Z_p$. Here p^{n_p} is the divisor of the conductor N corresponding to p. K-finiteness condition states that the right action of K on f generates a finite-dimensional vector space.

2. The representation functions are eigen functions of the Casimir operator C of $gl(2,R)$ with eigenvalue ρ so that irreducible representations of $gl(2,R)$ are obtained. An explicit representation of the Casimir operator is given by

$$C = \frac{X_0^2}{4} + X_+X - +X_-X_+ \ , \tag{4.2}$$

   where one has

$$X_0 \left(\begin{array}{cc} 0 & i \\ -i & 0 \end{array} \right) \ , \left(\begin{array}{cc} 1 & \mp i \\ \mp i & -1 \end{array} \right) \ . \tag{4.3}$$

3. The center $A^\times$ of $GL_e(2,A)$ consists of $A^\times$ multiples of identity matrix and it is assumed $f(gz) = \chi(z)f(g)$, where $\chi : A^\times \to C$ is a character providing a multiplicative representation of $A^\times$.

   item The so-called cuspidality condition is associated with the cusps. Planar cusp (`https://cutt.ly/sBxd9sH`) corresponds geometrically to a sharp tip. Derivatives of $x(t)$ and $y(t)$ with respect to parameter t become zero at cusp. The direction of the curve changes at the cusp. $x \geq 0$. Cusp catastrophe $x^3 - y^2 = 0$ provides a simple example. The tip of the cusp is added in the compactification of the hyperbolic 2-manifold defined by the space $\Gamma \backslash H^2$.

   The cuspidality condition

$$\int_{Q \backslash NA} f(\left(\begin{array}{cc} 1 & u \\ 0 & 1 \end{array} \right) g) du = 0 \tag{4.4}$$

   is satisfied [A4]. Note that the integration measure is adelic. Note also that the transformations appearing in integrand are an adelic generalization of the 1-parameter subgroup of Lorentz transformations leaving invariant light-like vector. The condition implies that the modular functions defined by the representation vanish at cusps at the boundaries of fundamental domains representing copies $H_u/\Gamma_0(N)$, where N is so called conductor. The "basic" cusp corresponds to $\tau = i\infty$ for the "basic" copy of the fundamental domain.

   The groups $gl(2,R)$, $O(2)$ and $GL_e(2,Q_p)$ act non-trivially in these representations and it can be shown that a direct sum of irreps of $GL_e(2,A_F) \times gl(2,R)$ results with each irrep occurring only once. These representations are known as cuspidal automorphic representations.

The representation space for an irreducible cuspidal automorphic representation π is tensor product of representation spaces associated with the factors of the adele. To each factor one can assign ground state which is for un-ramified prime invariant under $Gl_2(Z_p)$ and in ramified case under $\Gamma_0(N)$. This ground states is somewhat analogous to the ground state of infinite-dimensional Fock space.

2. From adeles to $\Gamma_0(N)\backslash SL_e(2,R)$

The path from adeles to the modular forms in upper half plane involves many twists.

1. By so called central approximation theorem the group $GL_e(2,Q)\backslash GL_e(2,A)/K$ is isomorphic to the group $\Gamma_0(N)\backslash GL_+(2,R)$, where N is so called conductor, which is an integer measuring the ramification of the extension [A4] (`https://cutt.ly/DBcgOA2`). This means enormous simplification since one gets rid of the adelic factors altogether. Intuitively the reduction corresponds to the possibility to interpret rational number as collection of infinite number of p-adic rationals coming as powers of primes so that the element of $\Gamma_0(N)$ has interpretation also as Cartesian product of corresponding p-adic elements.

2. The group $\Gamma_0(N) \subset SL_e(2,Z)$ consists of matrices

$$\left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \ , \ c \ mod \ N = 0. \tag{4.5}$$

   $_+$ refers to positive determinant. Note that $\Gamma_0(N)$ contains as a subgroup congruence subgroup $\Gamma(N)$ consisting of matrices, which are unit matrices modulo N. Congruence subgroup is a normal subgroup of $SL_e(2,Z)$ so that also $SL_e(2,Z)/\Gamma_0(N)$ is group. Physically modular group $\Gamma(N)$ would be rather interesting alternative for $\Gamma_0(N)$ as a compact subgroup and the replacement $K_p = \Gamma_0(p^{k_p}) \to \Gamma(p^{k_p})$ of p-adic groups adelic decomposition is expected to guarantee this.

3. Central character condition together with assumptions about the action of K implies that the smooth functions in the original space (smoothness means local constancy in p-adic sectors: does this mean p-adic pseudo constancy?) are completely determined by their restrictions to $\Gamma_0(N)\backslash SL_e(2,R)$ so that one gets rid of the adeles.

3. From $\Gamma_0(N)\backslash SL_e(2,R)$ *to upper half-plane* $H_u = SL_e(2,R)/SO(2)$

The representations of $(gl(2,C), O(2))$ come in four categories corresponding to principal series, discrete series, the limits of discrete series, and finite-dimensional representations [A4]. For the discrete series representation π giving square integrable representation in $SL_e(2,R)$ one has $\rho = k(k-1)/4$, where $k > 1$ is integer. As sl_2 module, π_∞ is direct sum of irreducible Verma modules with highest weight $-k$ and lowest weight k. The former module is generated by a unique, up to a scalar, highest weight vector v_∞ such that

$$X_0 v_\infty = -k v_\infty \ \ , \ \ X_+ v_\infty = 0 \ \ . \tag{4.6}$$

The latter module is in turn generated by the lowest weight vector

$$\left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) v_\infty \ \ . \tag{4.7}$$

This means that entire module is generated from the ground state v_∞, and one can focus to the function ϕ_π on $\Gamma_0(N)\backslash SL_e(2,R)$ corresponding to this vector. The goal is to assign to this function $SO(2)$ invariant function defined in the upper half-plane $H_u = SL_e(2,R)/SO(2)$, whose points can be parameterized by the numbers $\tau = (a+bi)/(c+di)$ determined by $SL_e(2,R)$ elements. The function $f_\pi(g) = \phi_\pi(g)(ci+d)^k$ indeed is SO(2) invariant since the phase $exp(ik\phi)$ resulting in $SO(2)$ rotation by ϕ is compensated by the phase resulting from $(ci+d)$ factor. This function is not anymore $\Gamma_0(N)$ invariant but transforms as

$$f_\pi((a\tau+b)/(c\tau+d)) = (c\tau+d)^k f_\pi(\tau) \tag{4.8}$$

under the action of $\Gamma_0(N)$ The highest weight condition $X_+ v_\infty$ implies that f is holomorphic function of τ. Such functions are known as modular forms of weight k and level N. It would seem that the replacement of $\Gamma_0(N)$ suggested by physical arguments would only replace $H_u/\Gamma_0(N)$ with $H_u/\Gamma(N)$.

f_π can be expanded as power series in the variable $q = exp(2\pi\tau)$ to give

$$f_\pi(q) = \sum_{n=0}^{\infty} a_n q^n \ . \tag{4.9}$$

Cuspidality condition means that f_π vanishes at the cusps of the fundamental domain of the action of $\Gamma_0(N)$ on H_u. In particular, it vanishes at $q = 0$, which corresponds to $\tau = -\infty$. This implies $a_0 = 0$. This function contains all information about automorphic representation.

4.1.4 Hecke operators

Wikipedia provides a brief description of Hecke operators (`https://cutt.ly/hBxd5Yb`).

1. Spherical Hecke algebra (, which must be distinguished from non-commutative Hecke algebra associated with braids) can be defined as algebra of $GL_e(2, Z_p)$ bi-invariant functions on $GL_e(2, Q_p)$ with respect to convolution product. Sub-algebra of group algebra is in question.

2. This algebra is isomorphic to the polynomial algebra in two generators $H_{1,p}$ and $H_{2,p}$ and the ground states v_p of automorphic representations are eigenstates of these operators.

3. The normalizations can be chosen so that the second eigenvalue equals unity. Second eigenvalue must be an algebraic number. The eigenvalues of Hecke operators $H_{p,1}$ correspond to the coefficients a_p of the q-expansion of automorphic function f_π so that f_π is completely determined once these coefficients carrying number theoretic information are known [A4].

4. The action of Hecke operators induces an action on the modular function in the upper half-plane so that Hecke operators also have a representation as what is known as classical Hecke operators. The existence of this representation suggests that adelic representations might not be absolutely necessary for the realization of Langlands program.

From the TGD point of view a possible interpretation of this picture is in terms of modular invariance. Teichmüller parameters of the algebraic Riemann surface are affected by the absolute Galois group. This induces $Sl(2g, Z)$ transformation if the action does not change the conformal equivalence class and a more general transformation when it does. In the Gl_2 case discussed above one has $g = 1$ (torus). This change would correspond to non-trivial cuspidality conditions implying that ground state is invariant only under subgroups of $Gl_2(Z_p)$ for some primes. These primes would correspond to ramified primes in maximal Abelian extension of rationals.

An interesting possibility is that these representations can be continued from the hyperbolic 2-manifolds to hyperbolic 3-manifolds assignable to the mass shells H^3 defined by tessellations. The discrete subgroup Γ of $SL(2, R)$ would be continued to a discrete complex subgroup of $SL(2, C)$. There would be left invariance with respect to diagonal $SL(2, C)$. Finite sums over right translates by discrete elements of adelic $SL(2, C)$. Central character associated with Z_2. One could have a holography in the sense that the modular forms associated with the hyperbolic 2-manifold as boundary of hyperbolic manifold would be continued to their counterparts if 3-D hyperbolic manifold.

4.2 Some number theoretic notions

4.2.1 Frobenius automorphism

Frobenius automorphism `https://cutt.ly/NBkIudF` maps the element of a finite field $F(p, n)$, or more generally, of a commutative ring with characteristic p, to its p:th power and can there be regarded as an element of Galois group for an extension of finite field. F maps products to products and sums to sums.

For a finite field one has $x^p = x$ by Fermat's little theorem. The elements of F_p determined the roots of the equation $X^p = X$. There are no more roots in any extension. Therefore, if L is an algebraic extension of F_p, F_p is the fixed field of the Frobenius automorphism of L. The Galois group of an extension of a finite field is generated by the iterates of Frobenius automorphism.

4.2.2 The notion of discriminant

The discriminant of the polynomial is the most concrete definition (https://cutt.ly/GBxfyIm).

1. For a polynomial $P(x) = a_n x^n + ...$ the discriminant can be defined by the formula

$$\mathrm{Disc}_x(A) = a_n^{2n-2} \prod_{i<j} (r_i - r_j)^2 = (-1)^{n(n-1)/2} a_n^{2n-2} \prod_{i \neq j} (r_i - r_j) \ , \qquad (4.10)$$

   This notion applies to extensions of rationals defined by polynomials. For a second order polynomial $ax^2 + bx + x$, one has the familiar formula $\mathrm{Disc} = b^2 - 4ac$.

2. In the recent case the coefficients are rational. D vanishes when the polynomial has two or more identical roots which occurs for suitable values of parameters. The geometric interpretation is that two sheets (roots) of the graph of a root as a many-valued function of parameters a_i coincide so that the tangent space of the graph is parallel to x. Cusp catastrophe associated with a polynomial of order 3 is the simplests non-trivial example.

3. For a rational polynomials D is a rational number and for the ramified primes dividing D, it vanishes for the finite field variants of the polynomial with coefficients taken modulo p so that there are multiple roots for ramified primes. One can say that p-adically a catastrophe occurs in order $O(p) = 0$. This defines a p-adic variant of quantum criticality and gives an idea about the special physical role of the ramified primes in TGD.

A more abstract definition of the discriminant, which does not depend on the polynomial (https://cutt.ly/6BxfoQo). One distinguishes between the absolute discriminant of a number field and the relative discriminant of an extension of a number field. In the TGD framework, both situations are the same since number fields are extensions of rationals or induced by them.

1. One starts directly from the extension of rationals and imbeds the roots as complex numbers to plane. There is a large number of different embeddings. This corresponds to the fact that many polynomials P define the same extension. The counterpart for this non-uniqueness is that any basis elements for the basis for the ring of integers of the extension can define the unit to which the real axis is assigned.

2. There are n choices corresponding to n basic vectors of the integer basis consisting of algebraic integers, which are roots of a monic polynomial. One can choose the monic polynomial so that it is of degree n and the powers of a root define integer basis. Each choice s_i defines a map of the basis vectors e_j to the complex plane. The image vectors $s_i(e_j)$ define a matrix, whose determinant defines the discriminant D of the extension, which is the same as given by the less abstract definition based on the roots of a polynomial.

4.2.3 The notions of valuation and ramification

The notions of valuation and ramification (https://cutt.ly/bBgb47p) are easiest to understand in terms of a concrete polynomial representation of extension.

The extension with a given Galois group is obtained in very many ways. For instance, all irreducible polynomials of degree 2 have the same Galois group. Further information comes from the concrete polynomial representation. Ramified primes appear in the discrimant D of P as factors. For ramified primes, the splitting to a product of powers $\mathfrak{p}_i^{e_i}$ of prime ideals $\mathfrak{p}_i$ of extension is such that at least $e_i > 1$ appears. The discriminant is product for the squares of the differences of roots and depends on polynomial. This provides a more precice characterization of the situation than mere Galois group.

Ramified primes are special in the sense that for them the extension of p-adic number field induced by the extension of rationals is has lower dimension than for unramified primes. This is intuitively understandable since the discrimant vanishes in order $O(p)$ at least for the ramified prime. The prime ideals of K can split in to prime ideals of L. Also powers of primes of extension can appear in the splitting and this correspond to ramification. Ramified primes appear as factors in the discriminant.

The extension defined by a polynomials define a basis of algebraic integers and one can define norm by the determinant of the linear transformation defined by multiplication with an integer of the extension. This norm depends on the polynomial P and defines p-adic norm. The logarithm of the norm defines the valuation. When ramification occurs the dimension of p-adic extension l/k restricted to the finite field parts of p-adic numbers is lower than the dimension of extension L/K of rationals. The dimension of the corresponding finite field is lower than that for rationals.

In the abstract approach one does not mention polynomials at all and considers only valuations as norms assigned to an abstract extension of rationals. The equivalence class of valuations replaces the equivalence class of polynomials with the same Galois group and same discriminant if valuation is determined by the powers of ramified primes appearing in the discrimant.

Intuitively, the valuation should correspond to a prime ideal $\mathfrak{p}$ of L and to a norm. For extensions of rationals these prime ideals correspond to the primes defining extensions of p-adic number fields and these primes are special. Ramified primes are those appearing in the discriminant. The catastrophe theoretic picture based on the discriminant of the polynomial defining the catastrophe gives an idea of what is involved. This intuitive helps to make sense of the rather abstract statements below.

1. If there are several prime ideals, there are several valuations, which need not be equivalent (transform to each other by the action of Galois group). This would suggest that G_w transforms to each other prime ideals $\mathfrak{p}$ defining the same evaluation. Valuation ring R_w corresponds to the ring, whose elements have a non-negative norm or equivalenty, a given element x of O or its inverse belongs R_w. Is the valuation ring same as the ring formed by non-negative powers of this prime ideal? Valuation ring has maximal ideal m_w. The maximal ideal m_w of R_w representing the equivalence class of valuation inside the evaluation ring R_w is a key concept.

2. The ramification is characterized using decomposition group G_w and the hierarchy of ramification subgroups, which are normal subgroups of G_w. The decomposition group G_w of a valuation, which is determined by element w, is the subgroup of Galois group acting as the stabilizer group leaving the evaluation invariant.

   G_w must leave invariant the determinant defining the norm. How does G_w relate to the isotropy group of a given root of P? If G_w and the isotropy group are identical and the isotropy group depends on the root, a given polynomial P could allow several evaluations. If the maximal (prime) ideal p of $O(L)$ defines the extension, G_w would transform it to a prime defining an equivalent norm. By Hensel's lemma, the ring of $O(L)$ of L-integsds can be written as $O(L) = O_K(\alpha)$ for some α in $O(L)$.

3. The inertia group I_w of w consists of the elements of Galois group, which leave the elements of R_w invariant modulo m_w. These elements are analogous to p-adic integers numbers smaller than p and the intuitive picture is that ramification means that the generating element of the ring R_w is power of w which is larger than 1.

   Also the functional decomposition of polynomial P defines a hierarchey of normal subgroups as Galois subgroups and factor groups. Hierarchy of ramification groups must correspond to polynomials in a composition of P to polynomials.

   The inertia group of a given equivalence class of valuations is a subgroup of G_w and the stabilizer group of the valuation. It could correspond to the Galois group of the extension E_n associated with $P = P_n \circ ... \circ P_1$ regarded as an extension of the extension E_{n-1} associated with $P_{n-1} \circ ...P_1$.

4. There are also higher normal subgroups in a series associated with Gal. They give additional information about the valuation.

Also the notion of the conductor is involved. The conductor of an extension is an integer serving as measure for the ramification. Qualitatively, the extension is unramified if, and only if, the conductor is zero, and it is tamely ramified if, and only if, the conductor is 1. More precisely, the conductor computes the non-triviality of higher ramification groups. The description of conductor given in the Wikipedia article (`https://cutt.ly/DBcgOA2`) is extremely general and therefore too technical to be understood by a non-specialist.

4.2.4 Artin L-function

Given representation ρ of the Galois group G of the finite extension L/K on a finite-dimensional complex vector space V, the Artin L-function: $L(\rho, s)$ is defined by an Euler product. For each prime ideal $\mathfrak{p}$ in K's ring of integers, there is an Euler factor, which is easiest to define in the case where $\mathfrak{p}$ is unramified in L (true for almost all $\mathfrak{p}$).

In that case, the Frobenius element $\mathbf{Frob}(\mathfrak{p})$ mapping elements of the ring of integers of the extension L/K to its p:th power is identified as a conjugacy class in G. Therefore, the characteristic polynomial of $\rho(\mathbf{Frob}(\mathfrak{p}))$ is well-defined. The Euler factor for $\mathfrak{p}$ is a slight modification of the characteristic polynomial, equally well-defined,

$$\mathrm{charpoly}(\rho(\mathbf{Frob}(\mathfrak{p})))^{-1} = \det\left[I - t\rho(\mathbf{Frob}(\mathfrak{p}))\right]^{-1} \quad , \tag{4.11}$$

as rational function in t, evaluated at

$$t = N(\mathfrak{p})^{-s} \quad , \tag{4.12}$$

with s a complex variable in the usual Riemann zeta function notation. (Here N is the field norm of an ideal.)

When $\mathfrak{p}$ is ramified, and I is the inertia group which is a subgroup of G, a similar construction is applied, but to the subspace of V fixed (pointwise) by I.

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