TGD as a Generalized Number Theory III: Infinite Primes

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Contents

Abstract

Infinite primes are besides p-adicization and the representation of space-time region as a associative (co-associative) sub-manifold of hyper-octonionic space the basic pillars of the vision about TGD as a generalized number theory and will be discussed in the third part of the multi-chapter devoted to the attempt to articulate this vision as clearly as possible.

1. Why infinite primes are unavoidable

Suppose that 3-surfaces could be characterized by p-adic primes characterizing their effective p-adic topology. p-Adic unitarity implies that each quantum jump involves unitarity evolution U followed by a quantum jump. Simple arguments show that the p-adic prime characterizing the 3-surface representing the entire universe increases in a statistical sense. This leads to a peculiar paradox: if the number of quantum jumps already occurred is infinite, this prime is most naturally infinite. On the other hand, if one assumes that only finite number of quantum jumps have occurred, one encounters the problem of understanding why the initial quantum history was what it was. Furthermore, since the size of the 3-surface representing the entire Universe is infinite, p-adic length scale hypothesis suggest also that the p-adic prime associated with the entire universe is infinite.

These arguments motivate the attempt to construct a theory of infinite primes and to extend quantum TGD so that also infinite primes are possible. Rather surprisingly, one can construct what might be called generating infinite primes by repeating a procedure analogous to a quantization of a super symmetric quantum field theory. At given level of hierarchy one can identify the decomposition of space-time surface to p-adic regions with the corresponding decomposition of the infinite prime to primes at a lower level of infinity: at the basic level are finite primes for which one cannot find any formula.

2. Two views about the role of infinite primes and physics in TGD Universe

Two different views about how infinite primes, integers, and rationals might be relevant in TGD Universe have emerged.

1. The first speculative view is based on the idea that infinite primes characterize quantum states of the entire Universe. 8-D complexified octonions containing hyper-octonions as sub-space would make this correspondence very concrete since 8-D hyper-octonions have interpretation as 8-momenta. By quantum-classical correspondence also the decomposition of space-time surfaces to p-adic space-time sheets should be coded by infinite hyper-octonionic primes. Infinite primes could even have a representation as hyper-quaternionic 4-surfaces of 8-D hyper-octonionic imbedding space. This view is addmittedly speculative since the notion of octonionic prime makes sense only for complexified octonions.
2. The second view is based on the idea that infinitely structured space-time points define space-time correlates of mathematical cognition. The mathematical analog of Brahman=Atman identity would however suggest that both views deserve to be taken seriously.

3. Infinite primes and infinite hierarchy of second quantizations

The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. Later it became clear that the process generalizes so that it applies in the case of quaternionic and octonionic primes and their hyper counterparts. This hierarchy of second quantizations means an enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes. This hierarchy can be identified with the corresponding hierarchy of space-time sheets of the many-sheeted space-time.

One can even try to understand the quantum numbers of physical particles in terms of infinite primes. In particular, the hyper-quaternionic primes correspond four-momenta and mass squared is prime valued for them. The properties of 8-D hyper-octonionic primes motivate the attempt to identify the quantum numbers associated with CP_2 degrees of freedom in terms of these primes. The representations of color group $SU(3)$ are indeed labelled by two integers and the states inside given representation by color hyper-charge and iso-spin.

It turns out that associativity constraint allows only rational infinite primes. One can however replace classical associativity with quantum associativity for quantum states assigned with infinite prime. One can also decompose rational infinite primes to hyper-octonionic

infinite primes at lower level of the hierarchy. Physically this would mean that the number theoretic 8-momenta have only time-component. This decomposition is completely analogous to the decomposition of hadrons to its colored constituents and might be even interpreted in terms of color confinement. The interpretation of the decomposition of rational primes to primes in the algebraic extensions of rationals, hyper-quaternions, and hyper-octonions would have an interpretation as an increase of number theoretical resolution and the principle of number theoretic confinement could be seen as a fundamental physical principle implied by associativity condition.

4. Space-time correlates of infinite primes

Infinite primes code naturally for Fock states in a hierarchy of super-symmetric arithmetic quantum field theories. Quantum classical correspondence leads to ask whether infinite primes could also code for the space-time surfaces serving as symbolic representations of quantum states. This would a generalization of algebraic geometry would emerge and could reduce the dynamics of Kähler action to algebraic geometry and organize 4-surfaces to a physical hierarchy according to their algebraic complexity. Note that this conjecture should be consistent with two other conjectures about the dynamics of space-time surfaces (space-time surfaces as preferred extrema of Kähler action and space-time surfaces as quaternionic or co-quaternionic (as associative or co-associative) 4-surfaces of hyper-octonion space M^8).

The representation of space-time surfaces as algebraic surfaces in M^8 is however too naive idea and the attempt to map hyper-octonionic infinite primes to algebraic surfaces seems has not led to any concrete progress.

The endless updating of quantum TGD might be blamed to be a waste of time. The interaction of new ideas with old ones has however again and again turned out to be an extremely fruitful process leading to rather precise view about how infinite hyper-octonionic rationals can be mapped to space-time surfaces without ad hoc assumptions. The progress in quantum TGD during the second half of the first decade of the new millenium led to several new and quite convincing ideas. Mention only zero energy ontology, the generalization of the imbedding space concept realizing the hierarchy of Planck constants, hyper-finite factors and their inclusions, and in particular, the realization of quantum classical correspondence in terms of measurement interaction term associated with the Kähler-Dirac action.

The crucial observation is that quantum classical correspondence allows to map quantum numbers of WCW spinor fields to space-time geometry. Therefore, if one wants to map infinite rationals to space-time geometry it is enough to map infinite primes to quantum numbers. This map can be indeed achieved thanks to the detailed picture about the interpretation of the symmetries of infinite primes in terms of standard model symmetries.

5. Generalization of ordinary number fields: infinite primes and cognition

Both fermions and p-adic space-time sheets are identified as correlates of cognition in TGD Universe. The attempt to relate these two identifications leads to a rather concrete model for how bosonic generators of super-algebras correspond to either real or p-adic space-time sheets (actions and intentions) and fermionic generators to pairs of real space-time sheets and their p-adic variants obtained by algebraic continuation (note the analogy with fermion hole pairs).

The introduction of infinite primes, integers, and rationals leads also to a generalization of real numbers since an infinite algebra of real units defined by finite ratios of infinite rationals multiplied by ordinary rationals which are their inverses becomes possible. These units are not units in the p-adic sense and have a finite p-adic norm which can be differ from one. This construction generalizes also to the case of hyper- quaternions and -octonions although non-commutativity and in case of octonions also non-associativity pose technical problems. Obviously this approach differs from the standard introduction of infinitesimals in the sense that sum is replaced by multiplication meaning that the set of real and also more general units becomes infinitely degenerate.

Infinite primes form an infinite hierarchy so that the points of space-time and imbedding space can be seen as infinitely structured and able to represent all imaginable algebraic structures. Certainly counter-intuitively, single space-time point is even capable of representing the quantum state of the entire physical Universe in its structure. For instance, in the real sense surfaces in the space of units correspond to the same real number 1, and single point, which is structure-less in the real sense could represent arbitrarily high-dimensional spaces as unions of real units.

One might argue that for the real physics this structure is completely invisible and is relevant only for the physics of cognition. On the other hand, one can consider the possibility of mapping the WCW and WCW spinor fields to the number theoretical anatomies of a single point of imbedding space so that the structure of this point would code for the world of classical worlds and for the quantum states of the Universe. Quantum jumps would induce changes of WCW spinor fields interpreted as wave functions in the set of number theoretical anatomies of single point of imbedding space in the ordinary sense of the word, and evolution would reduce to the evolution of the structure of a typical space-time point in the system. Physics would reduce to space-time level but in a generalized sense. Universe would be an algebraic hologram, and there is an obvious connection both with Brahman=Atman identity of Eastern philosophies and Leibniz's notion of monad.

Infinite rationals are in one-one correspondence with quantum states and in zero energy ontology hyper-octonionic units identified as ratios of the infinite integers associated with the positive and negative energy parts of the zero energy state define a representation of WCW spinor fields. The action of subgroups of SU(3) and rotation group SU(2) preserving hyper-octonionic and hyper-quaternionic primeness and identification of momentum and electro-weak charges in terms of components of hyper-octonionic primes makes this representation unique. Hence Brahman-Atman identity has a completely concrete realization and fixes completely the quantum number spectrum including particle masses and correlations between various quantum numbers.

1 Introduction

The third part of the multi-chapter discussing the idea about physics as a generalized number theory is devoted to the possible role of infinite primes in TGD.

The notion of prime seems to capture something very essential about what it is to be elementary building block of matter and has become a fundamental conceptual element of TGD. The notion of prime gains it generality from its reducibility to the notion of prime ideal of an algebra. Thus the notion of prime is well-defined, not only in case of quaternions and octonions, but also for their complexifications and one can speak about infinite primes in the case of hyper-quaternions and -octonions, which are especially natural physically and for which numbers having zero norm correspond physically to light-like 8-vectors.

1.1 The Notion Of Infinite Prime

The original motivation for the notion of infinite prime came from the first attempts to construct TGD inspired theory of consciousness (around 1995) [K12]. Suppose very naively that the 4-surfaces in a given sector of the "world of classical worlds" (WCW) are labelled by a fixed p-adic prime. The natural expectation is that evolution by quantum jumps means dispersion in the space of these sectors and leads to the increase of the p-adic prime characterizing the Universe. As one moves backwards in subjective time (sequence of quantum jumps) one ends up to the situation in which the prime characterizing the universe was $p = 2$. Should one assume that there was the first quantum jump when everything began? If not, then it would seem that the p-adic prime characterizing the Universe must be infinite. Second problem is that the p-adic length scales are finite and if the size scale of Universe is given by p-adic length scale the Universe has finite sized: this does not make sense in TGD framework. The only way out of the problems is the assumption that the p-adic prime characterizing the entire Universe is literally infinite and that p-adic primes characterizing space-time sheets are finite.

These argument, which are by no means central for the recent view about p-adic primes, motivated the attempt to construct a theory of infinite primes and to extend quantum TGD accordingly. This turns out to be possible. The recipe for constructing infinite primes is structurally equivalent with a repeated second quantization of an arithmetic super-symmetric quantum field theory. At the lowest level one has fermionic and bosonic states labeled by finite primes and infinite primes correspond to many particle states of this theory. Also infinite primes analogous to bound states are predicted. This hierarchy of quantizations can be continued indefinitely by taking the many particle states of the previous level as elementary particles at the next level. It must be also emphasized that the notion of infinity is relativistic. With respect to the p-adic norm infinite primes have unit norm for all finite and infinite primes so that there is nothing to become scared of!

Construction could make sense also for hyper-quaternionic and hyper-octonionic primes although non-commutativity and non-associativity pose technical challenges. One can also construct infinite number of real units as ratios of infinite integers with a precise number theoretic anatomy. The fascinating finding is that the quantum states labeled by standard model quantum numbers allow a representation as wave functions in the discrete space of these units. Space-time point becomes infinitely richly structured in the sense that one can associate to it a wave function in the space of real (or octonionic) units allowing to represent the WCW spinor fields. One can speak about algebraic holography or number theoretic Brahman=Atman identity and one can also say that the points of imbedding space and space-time surface are subject to a number theoretic evolution. In philosophical mood one can of course also ask whether there exists a hierarchy of imbedding spaces in which the imbedding space at the lower level represents something with infinitesimal size in the sense of real topology and whether this hierarchy is accompanied also by a hierarchy of conscious entities.

This picture suggest that the Universe of quantum TGD might basically provide a physical representation of number theory allowing also infinite primes. The proposal suggests also a possible generalization of real numbers to a number system akin to hyper-reals introduced by Robinson in his non-standard calculus [A10] providing a rigorous mathematical basis for calculus. In fact, some rather natural requirements lead to a unique generalization for the concepts of integer, rational and real. Infinite integers and reals can be regarded as infinite-dimensional vector spaces with integer and real valued coefficients respectively. Same generalization could make sense for all classical number fields.

1.2 Infinite Primes And Physics In TGD Universe

Several different views about how infinite primes, integers, and rationals might be relevant in TGD Universe have emerged.

1.2.1 Infinite primes and super-symmetric quantum field theory

Consider next the physical interpretation.

1. The discovery of infinite primes suggested strongly the possibility to reduce physics to number theory. The construction of infinite primes can be regarded as a repeated second quantization of a super-symmetric arithmetic quantum field theory. This suggests that configuration space spinor fields or at least the ground states of associated super-conformal representations could be mapped to infinite primes in both bosonic and fermionic degrees of freedom. The process might generalize so that it applies in the case of quaternionic and octonionic primes and their hyper counterparts. This hierarchy of second quantizations means enormous generalization of physics to what might be regarded a physical counterpart for a hierarchy of abstractions about abstractions about.... The ordinary second quantized quantum physics corresponds only to the lowest level infinite primes.

2. The ordinary primes appearing as building blocks of infinite primes at the first level of the hierarchy could be identified as coding for p-adic primes assignable to fermionic and bosonic partons identified as 2-surfaces of a given space-time sheet. The hierarchy of infinite primes would correspond to hierarchy of space-time sheets defined by the topological condensate. This leads also to a precise identification of p-adic and real variants of bosonic partonic 2-surfaces as correlates of intention and action and pairs of p-adic and real fermionic partons as correlates for cognitive representations.

3. The idea that infinite primes characterize quantum states of the entire Universe, perhaps ground states of super-conformal representations, if not all states, could be taken further. It turns out that this idea makes sense when one considers discrete wave functions in the space of infinite primes and that one can indeed represent standard model quantum numbers in this manner.

4. The number theoretical supersymmetry suggests also space-time supersymmetry TGD framework. Space-time super-symmetry in its standard form is not possible in TGD Universe and this cheated me to believe that this supersymmetry is completely absent in TGD Universe. The progress in the understanding of the properties of the modified Dirac action however led

to a generalization of the space-time super-symmetry as a dynamical and broken symmetry of quantum TGD [K6].

Here however emerges the idea about the number theoretic analog of color confinement. Rational (infinite) primes allow not only a decomposition to (infinite) primes of algebraic extensions of rationals but also to algebraic extensions of quaternionic and octonionic (infinite) primes. The physical analog is the decomposition of a particle to its more elementary constituents. This fits nicely with the idea about number theoretic resolution represented as a hierarchy of Galois groups defined by the extensions of rationals and realized at the level of physics in terms of Jones inclusions [K13] defined by these groups having a natural action on space-time surfaces, induced spinor fields, and on configuration space spinor fields representing physical states [K2].

1.2.2 Infinite primes and physics as number theory

The hierarchy of algebraic extensions of rationals implying corresponding extensions of p-adic numbers suggests that Galois groups, which are the basic symmetry groups of number theory, should have concrete physical representations using induced spinor fields and configuration space spinor fields and also infinite primes and real units formed as infinite rationals. These groups permute zeros of polynomials and thus have a concrete physical interpretation both at the level of partonic 2-surfaces dictated by algebraic equations and at the level of braid hierarchy. The vision about the role of hyperfinite factors of II_1 and of Jones inclusions as descriptions of quantum measurements with finite measurement resolution leads to concrete ideas about how these groups are realized.

G_2 acts as automorphisms of hyper-octonions and $SU(3)$ as its subgroup respecting the choice of a preferred imaginary unit. The discrete subgroups of $SU(3)$ permuting to each other hyper-octonionic primes are analogous to Galois group and turned out to play a crucial role in the understanding of the correspondence between infinite hyper-octonionic primes and quantum states predicted by quantum TGD.

1.2.3 The notion of finite measurement resolution as the key concept

TGD predicts several hierarchies: the hierarchy of space-time sheets, the hierarchy of infinite primes, the hierarchy of Jones inclusions identifiable in terms of finite measurement resolution [K13]. the dark matter hierarchy characterized by increasing values of $\hbar$ [K5]. the hierarchy of extensions of a given p-adic number field. TGD inspired theory of consciousness predictes the hierarchy of selves and quantum jumps with increasing duration with respect to geometric time. These hierarchies should be closely related.

The notion of finite measurement resolution turns out to be the key concept: the p-adic norm of the rational defined by the infinite prime characterizes the angle measurement resolution for given p-adic prime p. It is essential that one has what might be called a state function reduction selecting a fixed p-adic prime which could be also infinite. This gives direct connections with cognition and with the p-adicization program relying also on angle measurement resolution. Also the value the integers characterizing the singular coverings of CD and CP_2 defining as their product Planck constant characterize the measurement resolution for a given p-adic prime in CD and CP_2 degrees of freedom. This conforms with the fact that elementary particles are characterized by two infinite primes. Hence finite measurement resolution ties tightly together the three threads of the number theoretic vision. Finite measurement resolution relates also closely to the inclusions of hyper-finite factors central for TGD inspired quantum measurement theory with finite measurement resolution.

1.2.4 Space-time correlates of infinite primes

Infinite primes code naturally for Fock states in a hierarchy of super-symmetric arithmetic quantum field theories. Quantum classical correspondence leads to ask whether infinite primes could also code for the space-time surfaces serving as symbolic representations of quantum states. This would a generalization of algebraic geometry would emerge and could reduce the dynamics of Kähler action to algebraic geometry and organize 4-surfaces to a physical hierarchy according to their algebraic complexity. This conjecture should be consistent with two other conjectures about the dynamics of space-time surfaces (space-time surfaces as preferred extrema of Kähler action and space-time surfaces as quaternionic or co-quaternionic (as associative or co-associative) 4-surfaces of hyper-octonion space M^8).

Quantum classical correspondence requires the map of the quantum numbers of configuration space spinor fields to space-time geometry. The Kähler-Dirac equation with measurement interaction term realizes this requirement. Therefore, if one wants to map infinite rationals to space-time geometry it is enough to map infinite primes to quantum numbers. This map can be indeed achieved thanks to the detailed picture about the interpretation of the symmetries of infinite primes in terms of standard model symmetries. The notion of finite measurement resolution allows to deduce much more detailed about this correspondence. In particular, the rational defined by the infinite prime classifies the finite sub-manifold geometry defined by the discretization of the partonic 2-surface implied by the finite measurement resolution. Also a direct correlation between integers defining Planck constant and the "fermionic" part of the infinite prime emerges.

1.3 Infinite Primes, Cognition, And Intentionality

The correlation of infinite primes with cognition is the first fascinating possibility and this possibility has stimulated several ideas.

1. One can define the notion of prime also for the algebraic extensions of rationals. The hierarchy of infinite primes associated with algebraic extensions of rationals leading gradually towards algebraic closure of rationals would in turn define cognitive hierarchy corresponding to algebraic extensions of p-adic numbers.

2. The introduction of infinite primes, integers, and rationals leads also to a generalization of classical number fields since an infinite algebra of real (complex, etc...) units defined by finite ratios of infinite rationals multiplied by ordinary rationals which are their inverses becomes possible. These units are not units in the p-adic sense and have a finite p-adic norm which can be differ from one. This construction generalizes also to the case of hyper- quaternions and -octonions although non-commutativity and in case of octonions also non-associativity pose technical problems. Obviously this approach differs from the standard introduction of infinitesimals in the sense that sum of infinitesimals (real zeros) is replaced by multiplication of real units meaning that the set of real and also more general units becomes infinitely degenerate.

3. Infinite primes form an infinite hierarchy so that the points of space-time and imbedding space can be seen as infinitely structured and able to represent all imaginable algebraic structures. Certainly counter-intuitively, single space-time point -or more generally wave functions in the space of the units associated with the point- might be even capable of representing the quantum state of the entire physical Universe in its structure. For instance, in the real sense surfaces in the space of units correspond to the same real number 1, and single point, which is structure-less in the real sense could represent arbitrarily high-dimensional spaces as unions of real units. For real physics this structure is completely invisible and is relevant only for the physics of cognition. One can say that Universe is an algebraic hologram, and there is an obvious connection both with Brahman=Atman identity of Eastern philosophies and Leibniz's notion of monad.

4. In zero energy ontology hyper-octonionic units identified as ratios of the infinite integers associated with the positive and negative energy parts of the zero energy state define a representation of WCW spinor fields. The action of subgroups of SU(3) and rotation group SU(2) preserving hyper-octonionic and hyper-quaternionic primeness and identification of momentum and electro-weak charges in terms of components of hyper-octonionic primes makes this representation unique. Hence Brahman-Atman identity has a completely concrete realization and fixes completely the quantum number spectrum including particle masses and correlations between various quantum numbers.

5. One can assign to infinite primes at n^{th} level of hierarchy rational functions of n rational arguments which form a natural hierarchical structure in that highest level corresponds to a polynomial with coefficients which are rational functions of the arguments at the lower level. One can solve one of the arguments in terms of lower ones to get a hierarchy of

algebraic extensions. At the lowest level algebraic extensions of rationals emerge, at the next level algebraic extensions of space of rational functions of single variable, etc... This would suggest that infinite primes code for the correlation between quantum states and the algebraic extensions appearing in their their physical description and characterizing their cognitive correlates. The hierarchy of infinite primes would also correlate with a hierarchy of logics of various orders (hierarchy of statements about statements about...).

The appendix of the book gives a summary about basic concepts of TGD with illustrations. Pdf representation of same files serving as a kind of glossary can be found at `http://tgdtheory.fi/tgdglossary.pdf` [L3].

2 Infinite Primes, Integers, And Rationals

The definition of the infinite integers and rationals is a straightforward procedure and structurally similar to a repeated second quantization of a super-symmetric quantum field theory but including also the number theoretic counterparts of bound states.

2.1 The First Level Of Hierarchy

In the following the concept of infinite prime is developed gradually by stepwise procedure rather than giving directly the basic definitions. The hope is that the development of the concept in the same manner as it actually occurred would make it easier to understand it.

Step 1

One could try to define infinite primes P by starting from the basic idea in the proof of Euclid for the existence of infinite number of primes. Take the product of all finite primes and add 1 to get a new prime:

$$\begin{aligned} P &= 1+X \ , \\ X &= \prod_p p \ . \end{aligned} \tag{2.1}$$

If P were divisible by finite prime then $P-X=1$ would be divisible by finite prime and one would encounter contradiction. One could of course worry about the possible existence of infinite primes smaller than P and possibly dividing P. The numbers $N=P-k$, $k>1$, are certainly not primes since k can be taken as a factor. The number $P'=P-2=-1+X$ could however be prime. P is certainly not divisible by $P-2$. It seems that one cannot express P and $P-2$ as product of infinite integer and finite integer. Neither it seems possible to express these numbers as products of more general numbers of form $\prod_{p\in U} p+q$, where U is infinite subset of finite primes and q is finite integer.

Step 2

P and $P-2$ are not the only possible candidates for infinite primes. Numbers of form

$$\begin{aligned} P(\pm,n) &= \pm 1+nX \ , \\ k(p) &= 0,1,..... \ , \\ n &= \prod_p p^{k(p)} \ , \\ X &= \prod_p p \ , \end{aligned} \tag{2.2}$$

where $k(p)\neq 0$ holds true only in finite set of primes, are characterized by a integer n, and are also good prime candidates. The ratio of these primes to the prime candidate P is given by integer n. In general, the ratio of two prime candidates $P(m)$ and $P(n)$ is rational number m/n telling which of the prime candidates is larger. This number provides ordering of the prime candidates $P(n)$. The reason why these numbers are good candidates for infinite primes is the same as above. No finite prime p with $k(p)\neq 0$ appearing in the product can divide these numbers since, by the same arguments as appearing in Euclid's theorem, it would divide also 1. On the other hand it seems difficult to invent any decomposition of these numbers containing infinite numbers. Already at this stage one can notice the structural analogy with the construction of multiboson states in quantum field theory: the numbers $k(p)$ correspond to the occupation numbers of bosonic states of quantum field theory in one-dimensional box, which suggests that the basic structure of QFT might have number theoretic interpretation in some very general sense. It turns out that this analogy generalizes.

Step 3

All $P(n)$ satisfy $P(n) \geq P(1)$. One can however also the possibility that $P(1)$ is not the smallest infinite prime and consider even more general candidates for infinite primes, which are smaller than $P(1)$. The trick is to drop from the infinite product of primes $X = \prod_p p$ some primes away by dividing it by integer $s = \prod_{p_i} p_i$, multiply this number by an integer n not divisible by any prime dividing s and to add to/subtract from the resulting number nX/s natural number ms such that m expressible as a product of powers of only those primes which appear in s to get

$$\begin{array}{l} P(\pm, m, n, s) = n\frac{X}{s} \pm ms \ , \\ m = \prod_{p|s} p^{k(p)} \ , \\ n = \prod_{p|\frac{X}{s}} p^{k(p)}, \ \ k(p) \geq 0 \ . \end{array} \tag{2.3}$$

Here $x|y$ means "x divides y". To see that no prime p can divide this prime candidate it is enough to calculate $P(\pm, m, n, s)$ modulo p: depending on whether p divides s or not, the prime divides only the second term in the sum and the result is nonzero and finite (although its precise value is not known). The ratio of these prime candidates to $P(+, 1, 1, 1)$ is given by the rational number n/s: the ratio does not depend on the value of the integer m. One can however order the prime candidates with given values of n and s using the difference of two prime candidates as ordering criterion. Therefore these primes can be ordered.

One could ask whether also more general numbers of the form $n\frac{X}{s} \pm m$ are primes. In this case one cannot prove the indivisibility of the prime candidate by p not appearing in m. Furthermore, for $s \ mod \ 2 = 0$ and $m \ mod \ 2 \neq 0$, the resulting prime candidate would be even integer so that it looks improbable that one could obtain primes in more general case either.

Step 4

An even more general series of candidates for infinite primes is obtained by using the following ansatz which in principle is contained in the original ansatz allowing infinite values of n

$$\begin{array}{l} P(\pm, m, n, s|r) = nY^r \pm ms \ , \\ Y = \frac{X}{s} \ , \\ m = \prod_{p|s} p^{k(p)} \ , \\ n = \prod_{p|\frac{X}{s}} p^{k(p)}, \ \ k(p) \geq 0 \ . \end{array} \tag{2.4}$$

The proof that this number is not divisible by any finite prime is identical to that used in the previous case. It is not however clear whether the ansatz for given r is not divisible by infinite primes belonging to the lower level. A good example in $r = 2$ case is provided by the following unsuccessful ansatz

$$\begin{array}{l} N = (n_1Y + m_1s)(n_2Y + m_2s) = \frac{n_1n_2X^2}{s^2} - m_1m_2s^2 \ , \\ Y = \frac{X}{s} \ , \\ n_1m_2 - n_2m_1 = 0 \ . \end{array}$$

Note that the condition states that n_1/m_1 and $-n_2/m_2$ correspond to the same rational number or equivalently that (n_1, m_1) and (n_2, m_2) are linearly dependent as vectors. This encourages the guess that all other $r = 2$ prime candidates with finite values of n and m at least, are primes. For higher values of r one can deduce analogous conditions guaranteeing that the ansatz does not reduce to a product of infinite primes having smaller value of r. In fact, the conditions for primality state that the polynomial $P(n, m, r)(Y) = nY^r + m$ with integer valued coefficients ($n > 0$) defined by the prime candidate is irreducible in the field of integers, which means that it does not reduce to a product of lower order polynomials of same type.

Step 5

A further generalization of this ansatz is obtained by allowing infinite values for m, which leads to the following ansatz:

$$\begin{array}{l} P(\pm,m,n,s|r_1,r_2) = nY^{r_1} \pm ms \ , \\ m = P_{r_2}(Y)Y + m_0 \ , \\ Y = \frac{X}{s} \ , \\ m_0 = \prod_{p|s} p^{k(p)} \ , \\ n = \prod_{p|Y} p^{k(p)}, \ \ k(p) \geq 0 \ . \end{array} \tag{2.5}$$

Here the polynomial $P_{r_2}(Y)$ has order r_2 is divisible by the primes belonging to the complement of s so that only the finite part m_0 of m is relevant for the divisibility by finite primes. Note that the part proportional to s can be infinite as compared to the part proportional to Y^{r_1}: in this case one must however be careful with the signs to get the sign of the infinite prime correctly. By using same arguments as earlier one finds that these prime candidates are not divisible by finite primes. One must also require that the ansatz is not divisible by lower order infinite primes of the same type. These conditions are equivalent to the conditions guaranteeing the polynomial primeness for polynomials of form $P(Y) = nY^{r_1} \pm (P_{r_2}(Y)Y + m_0)s$ having integer-valued coefficients. The construction of these polynomials can be performed recursively by starting from the first order polynomials representing first level infinite primes: Y can be regarded as formal variable and one can forget that it is actually infinite number.

By finite-dimensional analogy, the infinite value of m means infinite occupation numbers for the modes represented by integer s in some sense. For finite values of m one can always write m as a product of powers of $p_i|s$. Introducing explicitly infinite powers of p_i is not in accordance with the idea that all exponents appearing in the formulas are finite and that the only infinite variables are X and possibly S (formulas are symmetric with respect to S and X/S). The proposed representation of m circumvents this difficulty in an elegant manner and allows to say that m is expressible as a product of infinite powers of p_i despite the fact that it is not possible to derive the infinite values of the exponents of p_i.

Summarizing, an infinite series of candidates for infinite primes has been found. The prime candidates $P(\pm,m,n,s)$ labeled by rational numbers n/s and integers m plus the primes $P(\pm,m,n,s|r_1,r_2)$ constructed as r_1: th or r_2: th order polynomials of $Y = X/s$: the latter ansatz reduces to the less general ansatz of infinite values of n are allowed.

One can ask whether the $p \ mod \ 4 = 3$ condition guaranteeing that the square root of -1 does not exist as a p-adic number, is satisfied for $P(\pm,m,n,s)$. $P(\pm,1,1,1) \ mod \ 4$ is either 3 or 1. The value of $P(\pm,m,n,s) \ mod \ 4$ for odd s on n only and is same for all states containing even/odd number of $p \ mod \ = 3$ excitations. For even s the value of $P(\pm,m,n,s) \ mod \ 4$ depends on m only and is same for all states containing even/odd number of $p \ mod \ = 3$ excitations. This condition resembles G-parity condition of Super Virasoro algebras. Note that either $P(+,m,n,s)$ or $P(-,m,n,s)$ but not both are physically interesting infinite primes ($2m \ mod \ 4 = 2$ for odd m) in the sense of allowing complex Hilbert space. Also the additional conditions satisfied by the states involving higher powers of X/s resemble to Virasoro conditions. An open problem is whether the analogy with the construction of the many-particle states in super-symmetric theory might be a hint about more deeper relationship with the representation of Super Virasoro algebras and related algebras.

It is not clear whether even more general prime candidates exist. An attractive hypothesis is that one could write explicit formulas for all infinite primes so that generalized theory of primes would reduce to the theory of finite primes.

2.2 Infinite Primes Form A Hierarchy

By generalizing using general construction recipe, one can introduce the second level prime candidates as primes not divisible by any finite prime p or infinite prime candidate of type $P(\pm,m,n,s)$ (or more general prime at the first level: in the following we assume for simplicity that these are the only infinite primes at the first level). The general form of these prime candidates is exactly the same as at the first level. Particle-analogy makes it easy to express the construction recipe. In present case "vacuum primes" at the lowest level are of the form

$$\begin{array}{l} \frac{X_1}{S} \pm S \ , \\ X_1 = X \prod_{P(\pm,m,n,s)} P(\pm,m,n,s) \ , \\ S = s \prod_{P_i} P_i \ , \\ s = \prod_{p_i} p_i \ . \end{array} \quad (2.6)$$

S is product or ordinary primes p and infinite primes $P_i(\pm,m,n,s)$. Primes correspond to physical states created by multiplying X_1/S (S) by integers not divisible by primes appearing S (X_1/S). The integer valued functions $k(p)$ and $K(p)$ of prime argument give the occupation numbers associated with X/s and s type "bosons" respectively. The non-negative integer-valued function $K(P) = K(\pm,m,n,s)$ gives the occupation numbers associated with the infinite primes associated with X_1/S and S type "bosons". More general primes can be constructed by mimicking the previous procedure.

One can classify these primes by the value of the integer $K_{tot} = \sum_{P|X/S} K(P)$: for a given value of K_{tot} the ratio of these prime candidates is clearly finite and given by a rational number. At given level the ratio P_1/P_2 of two primes is given by the expression

$$\frac{P_1(\pm,m_1,n_1,s_1K_1,S_1}{P_2(\pm,m_2,n_2,s_2,K,S_2)} = \frac{n_1s_2}{n_2s_1} \prod_{\pm,m,n,s} (\frac{n}{s})^{K_1^+(\pm,n,m,s)-K_2^+(\pm,n,m,s)} \ . \quad (2.7)$$

Here K_i^+ denotes the restriction of $K_i(P)$ to the set of primes dividing X/S. This ratio must be smaller than 1 if it is to appear as the first order term $P_1P_2 \to P_1/P_2$ in the canonical identification and again it seems that it is not possible to get all rationals for a fixed value of P_2 unless one allows infinite values of N expressed neatly using the more general ansatz involving higher power of S.

2.3 Construction Of Infinite Primes As A Repeated Quantization Of A Super-Symmetric Arithmetic Quantum Field Theory

The procedure for constructing infinite primes is very much reminiscent of the second quantization of an super-symmetric arithmetic quantum field theory in which single particle fermion and boson states are labeled by primes. In particular, there is nothing especially frightening in the particle representation of infinite primes: theoretical physicists actually use these kind of representations quite routinely.

1. The binary-valued function telling whether a given prime divides s can be interpreted as a fermion number associated with the fermion mode labeled by p. Therefore infinite prime is characterized by bosonic and fermionic occupation numbers as functions of the prime labeling various modes and situation is super-symmetric. X can be interpreted as the counterpart of Dirac sea in which every negative energy state state is occupied and $X/s \pm s$ corresponds to the state containing fermions understood as holes of Dirac sea associated with the modes labeled by primes dividing s.

2. The multiplication of the "vacuum" X/s with $n = \prod_{p|X/s} p^{k(p)}$ creates $k(p)$ "p-bosons" in mode of type X/s and multiplication of the "vacuum" s with $m = \prod_{p|s} p^{k(p)}$ creates $k(p)$ "p-bosons". in mode of type s (mode occupied by fermion). The vacuum states in which bosonic creation operators act, are tensor products of two vacuums with tensor product represented as sum

$$|vac(\pm)\rangle = |vac(\frac{X}{s})\rangle \otimes |vac(\pm s)\rangle \leftrightarrow \frac{X}{s} \pm s \quad (2.8)$$

obtained by shifting the prime powers dividing s from the vacuum $|vac(X)\rangle = X$ to the vacuum ± 1. One can also interpret various vacuums as many fermion states. Prime property follows directly from the fact that any prime of the previous level divides either the first or second factor in the decomposition $NX/S \pm MS$.

3. This picture applies at each level of infinity. At a given level of hierarchy primes P correspond to all the Fock state basis of all possible many-particle states of second quantized super-symmetric theory. At the next level these many-particle states are regarded as single particle states and further second quantization is performed so that the primes become analogous to the momentum labels characterizing various single-particle states at the new level of hierarchy.

4. There are two nonequivalent quantizations for each value of S due to the presence of $\pm$ sign factor. Two primes differing only by sign factor are like G-parity $+1$ and -1 states in the sense that these primes satisfy $P\ mod\ 4 = 3$ and $P\ mod\ 4 = 1$ respectively. The requirement that -1 does not have p-adic square root so that Hilbert space is complex, fixes G-parity to say $+1$. This observation suggests that there exists a close analogy with the theory of Super Virasoro algebras so that quantum TGD might have interpretation as number theory in infinite context. An alternative interpretation for the $\pm$ degeneracy is as counterpart for the possibility to choose the fermionic vacuum to be a state in which either all positive or all negative energy fermion states are occupied.

5. One can also generalize the construction to include polynomials of $Y = X/S$ to get infinite hierarchy of primes labeled by the two integers r_1 and r_2 associated with the polynomials in question. An entire hierarchy of vacuums labeled by r_1 is obtained. A possible interpretation of these primes is as counterparts for the bound states of quantum field theory. The coefficient for the power $(X/s)^{r_1}$ appearing in the highest term of the general ansatz, codes the occupation numbers associated with vacuum $(X/s)^{r_1}$. All the remaining terms are proportional to s and combine to form, in general infinite, integer m characterizing various infinite occupation numbers for the subsystem characterized by s. The additional conditions guaranteeing prime number property are equivalent with the primality conditions for polynomials with integer valued coefficients and resemble Super Virasoro conditions. For $r_2 > 0$ bosonic occupation numbers associated with the modes with fermion number one are infinite and one cannot write explicit formula for the boson number.

6. One could argue that the analogy with super-symmetry is not complete. The modes of Super Virasoro algebra are labeled by natural number whereas now modes are labeled by prime. This need not be a problem since one can label primes using natural number n. Also 8-valued spin index associated with fermionic and bosonic single particle states in TGD world is lacking (space-time is surface in 8-dimensional space). This index labels the spin states of 8-dimensional spinor with fixed chirality. One could perhaps get also spin index by considering infinite octonionic primes, which correspond to vectors of 8-dimensional integer lattice such that the length squared of the lattice vector is ordinary prime:

$$\sum_{k=1,\ldots,8} n_k^2 = \text{ prime } .$$

Thus one cannot exclude the possibility that TGD based physics might provide representation for octonions extended to include infinitely large octonions. The notion of prime octonion is well defined in the set of integer octonions and it is easy to show that the Euclidian norm squared for a prime octonion is prime. If this result generalizes then the construction of generalized prime octonions would generalize the construction of finite prime octonions. It would be interesting to know whether the results of finite-dimensional case might generalize to the infinite-dimensional context. One cannot exclude the possibility that prime octonions are in one-one correspondence with physical states in quantum TGD.

These observations suggest a close relationship between quantum TGD and the theory of infinite primes in some sense: even more, entire number theory and mathematics might be reducible to quantum physics understood properly or equivalently, physics might provide the representation of basic mathematics. Of course, already the uniqueness of the basic mathematical structure of quantum TGD points to this direction. Against this background the fact that 8-dimensionality of the imbedding space allows introduction of octonion structure (also p-adic algebraic extensions) acquires new meaning. Same is also suggested by the fact that the algebraic extensions of p-adic numbers allowing square root of real p-adic number are 4- and 8-dimensional.

What is especially interesting is that the core of number theory would be concentrated in finite primes since infinite primes are obtained by straightforward procedure providing explicit formulas for them. Repeated quantization provides also a model of abstraction process understood as construction of hierarchy of natural number valued functions about functions about...... At the first level infinite primes are characterized by the integer valued function $k(p)$ giving occupation numbers plus subsystem-complement division (division to thinker and external world!). At the next level prime is characterized in a similar manner. One should also notice that infinite prime at given level is characterized by a pair $(R = MN, S)$ of integers at previous level. Equivalently, infinite prime at given level is characterized by fermionic and bosonic occupation numbers as functions in the set of primes at previous level.

2.4 Construction In The Case Of An Arbitrary Commutative Number Field

The basic construction recipe for infinite primes is simple and generalizes even to the case of algebraic extensions of rationals. Let $K = Q(\theta)$ be an algebraic number field (see the Appendix of [K10] for the basic definitions). In the general case the notion of prime must be replaced by the concept of irreducible defined as an algebraic integer with the property that all its decompositions to a product of two integers are such that second integer is always a unit (integer having unit algebraic norm, see Appendix of [K10]).

Assume that the irreducibles of $K = Q(\theta)$ are known. Define two irreducibles to be equivalent if they are related by a multiplication with a unit of K. Take one representative from each equivalence class of units. Define the irreducible to be positive if its first non-vanishing component in an ordered basis for the algebraic extension provided by the real unit and powers of θ, is positive. Form the counterpart of Fock vacuum as the product X of these representative irreducibles of K.

The unique factorization domain (UFD) property (see Appendix of [K10]) of infinite primes does not require the ring O_K of algebraic integers of K to be UFD although this property might be forced somehow. What is needed is to find the primes of K; to construct X as the product of all irreducibles of K but not counting units which are integers of K with unit norm; and to apply second quantization to get primes which are first order monomials. X is in general a product of powers of primes. Generating infinite primes at the first level correspond to generalized rationals for K having similar representation in terms of powers of primes as ordinary rational numbers using ordinary primes.

2.5 Mapping Of Infinite Primes To Polynomials And Geometric Objects

The mapping of the generating infinite primes to first order monomials labeled by their rational zeros is extremely simple at the first level of the hierarchy:

$$P_{\pm}(m,n,s) = \frac{mX}{s} \pm ns \rightarrow x_{\pm} \pm \frac{m}{sn} \quad . \tag{2.9}$$

Note that a monomial having zero as its root is not obtained. This mapping induces the mapping of all infinite primes to polynomials.

The simplest infinite primes are constructed using ordinary primes and second quantization of an arithmetic number theory corresponds in one-one manner to rationals. Indeed, the integer $s = \prod_i p_i^{k_i}$ defining the numbers k_i of bosons in modes k_i, where fermion number is one, and the integer r defining the numbers of bosons in modes where fermion number is zero, are co-prime. Moreover, the generating infinite primes can be written as $(n/s)X \pm ms$ corresponding to the two vacua $V = X \pm 1$ and the roots of corresponding monomials are positive *resp.* negative rationals.

More complex infinite primes correspond sums of powers of infinite primes with rational coefficients such that the corresponding polynomial has rational coefficients and roots which are not rational but belong to some algebraic extension of rationals. These infinite primes correspond simply to products of infinite primes associated with some algebraic extension of rationals. Obviously the construction of higher infinite primes gives rise to a hierarchy of higher algebraic extensions.

It is possible to continue the process indefinitely by constructing the Dirac vacuum at the n: th level as a product of primes of previous levels and applying the same procedure. At the second level Dirac vacuum $V = X \pm 1$ involves X which is the product of all primes at previous levels and in the polynomial correspondence X thus correspond to a new independent variable. At the n: th level one would have polynomials $P(q_1|q_2|...)$ of q_1 with coefficients which are rational functions of q_2 with coefficients which are.... The hierarchy of infinite primes would be thus mapped to the functional hierarchy in which polynomial coefficients depend on parameters depending on....

At the second level one representation of infinite primes would be as algebraic curve resulting as a locus of $P(q_1|q_2) = 0$: this certainly makes sense if q_1 and q_2 commute. At higher levels the locus is a higher-dimensional surface.

One can speculate with possible connections to TGD physics. The degree n of the polynomial is its basic characterizer. Infinite primes corresponding to polynomials of degree $n > 1$ should correspond to bound states. On the other hand, the hierarchy of Planck constants suggests strongly the interpretation in terms of gravitational bound states. Could one identify $h_{eff}/h = n$ as the degree of the polynomial characterizing infinite prime?

2.6 How To Order Infinite Primes?

One can order the infinite primes, integers and rationals. The ordering principle is simple: one can decompose infinite integers to two parts: the "large" and the "small" part such that the ratio of the small part with the large part vanishes. If the ratio of the large parts of two infinite integers is different from one or their sign is different, ordering is obvious. If the ratio of the large parts equals to one, one can perform same comparison for the small parts. This procedure can be continued indefinitely.

In case of infinite primes ordering procedure goes like follows. At given level the ratios are rational numbers. There exists infinite number of primes with ratio 1 at given level, namely the primes with same values of N and same S with MS infinitesimal as compared to NX/S. One can order these primes using either the relative sign or the ratio of $(M_1S_1)/(M_2S_2)$ of the small parts to decide which of the two is larger. If also this ratio equals to one, one can repeat the process for the small parts of M_iS_i. In principle one can repeat this process so many times that one can decide which of the two primes is larger. Same of course applies to infinite integers and also to infinite rationals build from primes with infinitesimal MS. If NS is not infinitesimal it is not obvious whether this procedure works. If $N_iX_i/M_iS_i = x_i$ is finite for both numbers (this need not be the case in general) then the ratio $\frac{M_1S_1}{M_2S_2}\frac{(1+x_2)}{(1+x_1)}$ provides the needed criterion. In case that this ratio equals one, one can consider use the ratio of the small parts multiplied by $\frac{(1+x_2)}{(1+x_1)}$ of M_iS_i as ordering criterion. Again the procedure can be repeated if needed.

2.7 What Is The Cardinality Of Infinite Primes At Given Level?

The basic problem is to decide whether Nature allows also integers S, $R = MN$ represented as infinite product of primes or not. Infinite products correspond to subsystems of infinite size (S) and infinite total occupation number (R) in QFT analogy.

1. One could argue that S should be a finite product of integers since it corresponds to the requirement of finite size for a physically acceptable subsystem. One could apply similar argument to R. In this case the set of primes at given level has the cardinality of integers ($alef_0$) and the cardinality of all infinite primes is that of integers. If also infinite integers R are assumed to involve only finite products of infinite primes the set of infinite integers is same as that for natural numbers.

2. NMP is well defined in p-adic context also for infinite subsystems and this suggests that one should allow also infinite number of factors for both S and $R = MN$. Super symmetric analogy suggests the same: one can quite well consider the possibility that the total fermion number of the universe is infinite. It seems however natural to assume that the occupation numbers $K(P)$ associated with various primes P in the representations $R = \prod_P P^{K(P)}$ are finite but nonzero for infinite number of primes P. This requirement applied to the modes associated with S would require the integer m to be explicitly expressible in powers of $P_i|S$

($P_{r_2} = 0$) whereas all values of r_1 are possible. If infinite number of prime factors is allowed in the definition of S, then the application of diagonal argument of Cantor shows that the number of infinite primes is larger than $alef_0$ already at the first level. The cardinality of the first level is $2^{alef_0}2^{alef_0} == 2^{alef_0}$. The first factor is the cardinality of reals and comes from the fact that the sets S form the set of all possible subsets of primes, or equivalently the cardinality of all possible binary valued functions in the set of primes. The second factor comes from the fact that integers $R = NM$ (possibly infinite) correspond to all natural number-valued functions in the set of primes: if only finite powers $k(p)$ are allowed then one can map the space of these functions to the space of binary valued functions bijectively and the cardinality must be 2^{alef_0}. The general formula for the cardinality at given level is obvious: for instance, at the second level the cardinality is the cardinality of all possible subsets of reals. More generally, the cardinality for a given level is the cardinality for the subset of possible subsets of primes at the previous level.

2.8 How To Generalize The Concepts Of Infinite Integer, Rational And Real?

The allowance of infinite primes forces to generalize also the concepts concepts of integer, rational and real number. It is not obvious how this could be achieved. The following arguments lead to a possible generalization which seems practical (yes!) and elegant.

2.8.1 Infinite integers form infinite-dimensional vector space with integer coefficients

The first guess is that infinite integers N could be defined as products of the powers of finite and infinite primes.

$$N = \prod_k p_k^{n_k} = nM \ , \ n_k \geq 0 \ , \tag{2.10}$$

where n is finite integer and M is infinite integer containing only powers of infinite primes in its product expansion.

It is not however not clear whether the sums of infinite integers really allow similar decomposition. Even in the case that this decomposition exists, there seems to be no way of deriving it. This would suggest that one should regard sums

$$\sum_i n_i M_i$$

of infinite integers as infinite-dimensional linear space spanned by M_i so that the set of infinite integers would be analogous to an infinite-dimensional algebraic extension of say p-adic numbers such that each coordinate axes in the extension corresponds to single infinite integer of form $N = mM$. Thus the most general infinite integer N would have the form

$$N = m_0 + \sum m_i M_i \ . \tag{2.11}$$

This representation of infinite integers indeed looks promising from the point of view of practical calculations. The representation looks also attractive physically. One can interpret the set of integers N as a linear space with integer coefficients m_0 and m_i:

$$N = m_0|1\rangle + \sum m_i|M_i\rangle \ . \tag{2.12}$$

$|M_i\rangle$ can be interpreted as a state basis representing many-particle states formed from bosons labeled by infinite primes p_k and $|1\rangle$ represents Fock vacuum. Therefore this representation is analogous to a quantum superposition of bosonic Fock states with integer, rather than complex valued, superposition coefficients. If one interprets M_i as orthogonal state basis and interprets m_i as p-adic integers, one can define inner product as

$$\langle N_a, N_b\rangle = m_0(a)m_0(b) + \sum_i m_i(a)m_i(b) \ . \quad (2.13)$$

This expression is well defined p-adic number if the sum contains only enumerable number of terms and is always bounded by p-adic ultra-metricity. It converges if the p-adic norm of of m_i approaches to zero when M_i increases.

2.8.2 Generalized rationals

Generalized rationals could be defined as ratios $R = M/N$ of the generalized integers. This works nicely when M and N are expressible as products of powers of finite or infinite primes but for more general integers the definition does not look attractive. This suggests that one should restrict the generalized rationals to be numbers having the expansion as a product of positive and negative primes, finite or infinite:

$$N = \prod_k p_k^{n_k} = \frac{n_1 M_1}{nM} \ . \quad (2.14)$$

2.8.3 Generalized reals form infinite-dimensional real vector space

One could consider the possibility of defining generalized reals as limiting values of the generalized rationals. A more practical definition of the generalized reals is based on the generalization of the pinary expansion of ordinary real number given by

$$\begin{aligned} x &= \sum_{n\geq n_0} x_n p^{-n} \ , \\ x_n &\in \{0,..,p-1\} \ . \end{aligned} \quad (2.15)$$

It is natural to try to generalize this expansion somehow. The natural requirement is that sums and products of the generalized reals and canonical identification map from the generalized reals to generalized p-adcs are readily calculable. Only in this manner the representation can have practical value.

These requirements suggest the following generalization

$$\begin{aligned} X &= x_0 + \sum_N x_N p^{-N} \ , \\ N &= \sum_i m_i M_i \ , \end{aligned} \quad (2.16)$$

where x_0 and x_N are ordinary reals. Note that N runs over infinite integers which has *vanishing finite part*. Note that generalized reals can be regarded as infinite-dimensional linear space such that each infinite integer N corresponds to one coordinate axis of this space. One could interpret generalized real as a superposition of bosonic Fock states formed from single single boson state labeled by prime p such that occupation number is either 0 or infinite integer N with a vanishing finite part:

$$X = x_0|0\rangle + \sum_N x_N|N > \ . \quad (2.17)$$

The natural inner product is

$$\langle X, Y\rangle = x_0 y_0 + \sum_N x_N y_N \ . \quad (2.18)$$

The inner product is well defined if the number of N: s in the sum is enumerable and x_N approaches zero sufficiently rapidly when N increases. Perhaps the most natural interpretation of the inner product is as R_p valued inner product.

The sum of two generalized reals can be readily calculated by using only sum for reals:

$$X + Y = x_0 + y_0 + \sum_N (x_N + y_N) p^{-N} \quad , \tag{2.19}$$

The product XY is expressible in the form

$$XY = x_0 y_0 + x_0 Y + X y_0 + \sum_{N_1, N_2} x_{N_1} y_{N_2} p^{-N_1 - N_2} \quad , \tag{2.20}$$

If one assumes that infinite integers form infinite-dimensional vector space in the manner proposed, there are no problems and one can calculate the sums $N_1 + N_2$ by summing component wise manner the coefficients appearing in the sums defining N_1 and N_2 in terms of infinite integers M_i allowing expression as a product of infinite integers.

Canonical identification map from ordinary reals to p-adics

$$x = \sum_k x_k p^{-k} \quad \rightarrow \quad x_p = \sum_k x_k p^k \quad ,$$

generalizes to the form

$$x = x_0 + \sum_N x_N p^{-N} \quad \rightarrow \quad (x_0)_p + \sum_N (x_N)_p p^N \quad , \tag{2.21}$$

so that all the basic requirements making the concept of generalized real calculationally useful are satisfied.

There are several interesting questions related to generalized reals.

1. Are the extensions of reals defined by various values of p-adic primes mathematically equivalent or not? One can map generalized reals associated with various choices of the base p to each other in one-one manner using the mapping

$$X = x_0 + \sum_N x_N p_1^{-N} \quad \rightarrow \quad x_0 + \sum_N x_N p_2^{-N} \quad . \tag{2.22}$$

   The ordinary real norms of *finite* (this is important!) generalized reals are identical since the representations associated with different values of base p differ from each other only infinitesimally. This would suggest that the extensions are physically equivalent. It these extensions are not mathematically equivalent then p-adic primes could have a deep role in the definition of the generalized reals.

2. One can generalize previous formulas for the generalized reals by replacing the coefficients x_0 and x_i by complex numbers, quaternions or octonions so as to get generalized complex numbers, quaternions and octonions. Also inner product generalizes in an obvious manner. The 8-dimensionality of the imbedding space provokes the question whether it might be possible to regard the infinite-dimensional WCW , or rather, its tangent space, as a Hilbert space realization of the generalized octonions. This kind of identification could perhaps reduce TGD based physics to generalized number theory.

2.9 Comparison With The Approach Of Cantor

The main difference between the approach of Cantor and the proposed approach is that Cantor uses only the basic arithmetic concepts such as sum and multiplication and the concept of successor defining ordering of both finite and infinite ordinals. Cantor's approach is also purely set theoretic. The problems of purely set theoretic approach are related to the question what the statement "Set is Many allowing to regard itself as One" really means and to the fact that there is no obvious connection with physics.

The proposed approach is based on the introduction of the concept of prime as a basic concept whereas partial ordering is based on the use of ratios: using these one can recursively define partial ordering and get precise quantitative information based on finite reals. The ordering is only partial and there is infinite number of ratios of infinite integers giving rise to same real unit which in turn leads to the idea about number theoretic anatomy of real point.

The "Set is Many allowing to regard itself as One" is defined as quantum physicist would define it: many particle states become single particle states in the second quantization describing the counterpart for the construction of the set of subsets of a given set. One could also say that integer as such corresponds to set as "One" and its decomposition to a product of primes corresponds to the set as "Many". The concept of prime, the ultimate "One", has as its physical counterpart the concept of elementary particle understood in very general sense. The new element is the physical interpretation: the sum of two numbers whose ratio is zero correspond to completely physical finite-subsystem-infinite complement division and the iterated construction of the set of subsets of a set at given level is basically p-adic evolution understood in the most general possible sense and realized as a repeated second quantization. What is attractive is that this repeated second quantization can be regarded also as a model of abstraction process and actually the process of abstraction itself.

The possibility to interpret the construction of infinite primes either as a repeated bosonic quantization involving subsystem-complement division or as a repeated super-symmetric quantization could have some deep meaning. A possible interpretation consistent with these two pictures is based on the hypothesis that fermions provide a reflective level of consciousness in the sense that the 2^N element Fock basis of many-fermion states formed from N single-fermion states can be regarded as a set of all possible statements about N basic statements. Statements about whether a given element of set X belongs to some subset S of X are certainly the fundamental statements from the point of view of mathematics. Hence one could argue that many-fermion states provide cognitive representation for the subsets of some set. Single fermion states represent the points of the set and many-fermion states represent possible subsets.

3 Can One Generalize The Notion Of Infinite Prime To TheNon-Commutative And Non-Associative Context?

The notion of prime and more generally, that of irreducible, makes sense also in more general number fields and even algebras. The considerations of [K11] suggests that the notion of infinite prime should be generalized to the case of complex numbers, quaternions, and octonions as well as to their hyper counterparts which seem to be physically the most interesting ones [K11]. Also the hierarchy of infinite primes should generalize as also the representation of infinite primes as polynomials although associativity is expected to pose technical problems.

3.1 Quaternionic And Octonionic Primes And Their Hyper Counterparts

The loss of commutativity and associativity implies that the definitions of quaternionic and octonionic primes are not completely straightforward.

3.1.1 Basic facts about quaternions and octonions

Both quaternions and octonions allow both Euclidian norm and the Minkowskian norm defined as a trace of the linear operator defined by the multiplication with octonion. Minkowskian norm has the metric signature of $H = M^4 \times CP_2$ or $M^4_+ \times CP_2$ so that H can be regarded locally as an octonionic space if one uses octonionic representation for the gamma matrices [K11]. Both norms are a multiplicative and the notions of both quaternionic and octonionic prime are well defined despite non-associativity. Quaternionic and octonionic primes have length squared equal to rational prime.

In the case of quaternions different basis of imaginary units I, J, K are related by 3-dimensional rotation group and different quaternionic basis span a 3-dimensional sphere. There is 2-sphere of complex structures since imaginary unit can be any unit vector of imaginary 3-space.

A basis for octonionic imaginary units J, K, L, M, N, O, P can be chosen in many manners and fourteen-dimensional subgroup G_2 of the group $SO(7)$ of rotations of imaginary units is the group labeling the octonionic structures related by octonionic automorphisms to each other. It deserves to be mentioned that G_2 is unique among the simple Lie-groups in that the ratio of the square roots of lengths for long and short roots of G_2 Lie-algebra are in ratio $3:1$. For other Lie-groups this ratio is either 2: 1 or all roots have same length. The set of equivalence classes of the octonion structures is $SO(7)/G_2 = S^7$. In the case of quaternions there is only one equivalence class.

The group of automorphisms for octonions with a fixed imaginary part is $SU(3)$. The coset space $S^6 = G_2/SU(3)$ labels possible complex structures of the octonion space specified by a selection of a preferred imaginary unit. $SU(3)/U(2) = CP_2$ could be thought of as the space of octonionic structures giving rise to a given quaternionic structure with complex structure fixed. This can be seen as follows. The units $1, I$ are $SU(3)$ singlets whereas J, J_1, J_2 and K, K_1, K_2 form $SU(3)$ triplet and antitriplet. Under $U(2)$ J and K transform like objects having vanishing $SU(3)$ isospin and suffer only a $U(1)$ phase transformation determined by multiplication with complex unit I and are mixed with each other in orthogonal mixture. Thus $1, I, J, K$ is transformed to itself under $U(2)$.

3.1.2 Quaternionic and octonionic primes

Quaternionic primes with $p \ mod \ 4 = 1$ can correspond to (n_1, n_2) with n_1 even and n_2 odd or vice versa. For $p \ mod \ 4 = 3$ (n_1, n_2, n_3) with n_i odd is the minimal option. In this case there is however large number of primes having only two components: in particular, Gaussian primes with $p \ mod \ 4 = 1$ define also quaternionic primes. Purely real Gaussian primes with $p \ mod \ 4 = 3$ with norm $z\overline{z}$ equal to p^2 are not quaternionic primes, and are replaced with 3-component quaternionic primes allowing norm equal to p. Similar conclusions hold true for octonionic primes.

The reality condition for polynomials associated with Gaussian infinite primes requires that the products of generating prime and its conjugate are present so that the outcome is a real polynomial of second order.

3.1.3 Hyper primes

The notion of prime generalizes to hyper-quaternionic and octonionic case. The factorization $n_0^2 - n_3^2 = (n_0 + n_3)(n_0 - n_3)$ implies that any hyper-quaternionic and -octonionic prime has one particualr representative as $(n_0, n_3, 0, ...) = (n_3 + 1, n_3, 0, ...)$, $n_3 = (p-1)/2$ for $p > 2$. $p = 2$ is exceptional: a representation with minimal number of components is given by $(2, 1, 1, 0, ...)$.

Notice that the interpretation of hyper-quaternionic primes (or integers) as four-momenta implies that it is not possible to find rest system for them if one assumes the entire quaternionic prime as four-momentum: only a system where energy is minimum is possible. The introduction of a preferred hyper-complex plane necessary for several reasons- in particular for the possibility to identify standard model quantum numbers in terms of infinite primes- allows to identify the momentum of particle in the preferred plane as the first two components of the hyper prime in fixed coordinate frame. Note that this leads to a universal spectrum for mass squared.

For time like hyper-primes the momentum is always time like for hyper-primes. In this case it is possible to find a rest frame by applying a hyper-primeness preserving G_2 transformation so that the resulting momentum has no component in the preferred frame. As a matter fact, SU(3) rotation is enough for a suitable choice of SU(3). These transformations form a discrete subgroup of SU(3) since hyper-integer property must be preserved. Massless states correspond to a null norm for the corresponding hyper integer unless one allows also tachyonic hyper primes with minimal representatives $(n_3, n_3 - 1, 0, ...)$, $n_3 = (p-1)/2$. Note that Gaussian primes with $p \ mod 4 \ = 1$

are representable as space-like primes of form $(0, n_1, n_2, 0)$: $n_1^2 + n_2^2 = p$ and would correspond to genuine tachyons. Space-like primes with $p \ mod \ 4 = 3$ have at least 3 non-vanishing components which are odd integers.

The notion of "irreducible" (see Appendix of [K10]) is defined as the equivalence class of primes related by a multiplication with a unit and is more fundamental than that of prime. All Lorentz boosts of a hyper prime combine to form an irreducible. Note that the units cannot correspond to real particles in corresponding arithmetic quantum field theory.

If the situation for $p > 2$ is effectively 2-dimensional in the sense that it is always possible to transform the hyper prime to a 2-component form by multiplying it by a suitable unit representing Lorentz boost, the theory for time-like hyper primes effectively reduces to the 2-dimensional hyper-complex case when irreducibles are chosen to belong to H_2. The physical counterpart for the choice of H_2 would be the choice of the plane of longitudinal polarizations, or equivalently, of quantization axis for spin. This hypothesis is physically highly attractive since it would imply number theoretic universality and conform with the effective 2-dimensionality. Of course, the hyper-octonionic primes related by $SO(7,1)$ boosts need not represent physically equivalent states.

Also the rigorous notion of hyper primeness seems to require effective 2-dimensionality. If effective 2-dimensionality holds true, hyper integers have a decomposition to a product of hyper primes multiplied by a suitable unit. The representation is obtained by Lorentz boosting the hyper integer first to a 2-component form and then decomposing it to a product of hyper-complex primes.

3.2 Hyper-Octonionic Infinite Primes

The infinite-primes associated with hyper-octonions are the most natural ones physically because of the underlying Lorentz invariance. It is however not possible to interpret them as as 8-momenta with mass squared equal to prime. The proper identification of standard model quantum numbers will be discussed later.

The basic objections against (hyper-)quaternionic and (hyper-)octonionic infinite primes relate to the non-commutativity and non-associativity.

In the case of quaternionic infinite primes non-commutativity, and in the case of octonionic infinite primes also non-associativity, might be expected to cause difficulties in the definition of X. Fortunately, the fact that all conjugates of a given finite prime appear in the product defining X, implies that the contribution from each irreducible with a given norm p is real and X is real. Therefore the multiplication and division of X with quaternionic or octonionic primes is a well-defined procedure, and generating infinite primes are well-defined apart from the degeneracy due to non-commutativity and non-associativity of the finite number of lower level primes.

Also the products of infinite primes are well defined, since by the reality of X it is possible to tell how the products AB and BA differ. Of course, also infinite primes representing physical states containing infinite numbers of fermions and bosons are possible and infinite primes of this kind must be analogous to generators of a free algebra for which AB and BA are not related in any manner.

Stronger form of associativity and commutativity is obtained if infinite octonionic/quaternionic primes are just ordinary octonionic/quaternonic primes multiplied with ordinary infinite primes. This option is perhaps the more elegant one. For this option the non-commutativity and non-associativity are concentrated on the finite octonionic/quaternionic prime multiplying the commutative infinite prime. This picture allows also the map of infinite octonionic/quaternionic primes to products of finite octonionic/quaternionic primes and of polynomials.

4 How To Interpret The Infinite Hierarchy Of Infinite Primes?

From the foregoing it should be clear that infinite primes might play key role in quantum physics. One can even consider the possibility that physics reduces to a generalized number theory, and that infinite primes are crucial for understanding mathematically consciousness and cognition. Of course, one must leave open the question whether infinite primes really provide really the mathematics of consciousness or whether they are only a beautiful but esoteric mathematical construct. In this spirit the following subsections give only different points of view to the problem with no attempt to a coherent overall view.

4.1 Infinite Primes And Hierarchy Of Super-Symmetric Arithmetic Quantum Field Theories

Infinite primes are a generalization of the notion of prime. They turn out to provide number theoretic correlates of both free, interacting and bound states of a super-symmetric arithmetic quantum field theory. It turns also possible to assign to infinite prime space-time surface as a geometric correlate although the original proposal for how to achieve this failed. Hence infinite primes serve as a bridge between classical and quantum and realize quantum classical correspondence stating that quantum states have classical counterparts, and has served as a basic heuristic guideline of TGD. More precisely, the natural hypothesis is that infinite primes code for the ground states of super-symplectic representations (for instance, ordinary particles correspond to states of this kind).

4.1.1 Generating infinite primes as counterparts of Fock states of a super-symmetric arithmetic quantum field theory

The basic construction recipe for infinite primes is simple and generalizes to the quaternionic case.

1. Form the product of all primes and call it X:

$$X = \prod_p p \ \ .$$

2. Form the vacuum states

$$V_{\pm} = X \pm 1 \ \ .$$

3. From these vacua construct all *generating* infinite primes by the following process. Kick out from the Dirac sea some negative energy fermions: they correspond to a product s of first powers of primes: $V \rightarrow X/s \pm s$ (s is thus square-free integer). This state represents a state with some fermions represented as holes in Dirac sea but no bosons. Add bosons by multiplying by integer r, which decomposes into parts as $r = mn$: m corresponding to bosons in X/s is product of powers of primes dividing X/s and n corresponds to bosons in s and is product of powers of primes dividing s. This step can be described as $X/s \pm s \rightarrow mX/s \pm ns$.

Generating infinite primes are thus in one-one correspondence with the Fock states of a super-symmetric arithmetic quantum field theory and can be written as

$$P_{\pm}(m, n, s) = \frac{mX}{s} \pm ns \ \ ,$$

where X is product of all primes at previous level. s is square free integer. m and n have no common factors, and neither m and s nor n and X/s have common factors.

The physical analog of the process is the creation of Fock states of a super-symmetric arithmetic quantum field theory. The factorization of s to a product of first powers of primes corresponds to many-fermion state and the decomposition of m and n to products of powers of prime correspond to bosonic Fock states since p^k corresponds to k-particle state in arithmetic quantum field theory.

4.1.2 More complex infinite primes as counterparts of bound states

Generating infinite primes are not all that are possible. One can construct also polynomials of the generating primes and under certain conditions these polynomials are non-divisible by both finite primes and infinite primes already constructed. As found, the conjectured effective 2-dimensionality for hyper-octonionic primes allows the reduction of polynomial representation of hyper-octonionic primes to that for hyper-complex primes. This would be in accordance with the effective 2-dimensionality of the basic objects of quantum TGD.

The physical counterpart of n: th order irreducible polynomial is as a bound state of n particles whereas infinite integers constructed as products of infinite primes correspond to non-bound but interacting states. This process can be repeated at the higher levels by defining the vacuum state to be the product of all primes at previous levels and repeating the process. A repeated second quantization of a super-symmetric arithmetic quantum field theory is in question.

The infinite primes represented by irreducible polynomials correspond to quantum states obtained by mapping the superposition of the products of the generating infinite primes to a superposition of the corresponding Fock states. If complex rationals are the coefficient field for infinite integers, this gives rise to states in a complex Hilbert space and irreducibility corresponds to a superposition of states with varying particle number and the presence of entanglement. For instance, the superpositions of several products of type $\prod_{i=1,..,n} P_i$ of n generating infinite primes are possible and in general give rise to irreducible infinite primes decomposing into a product of infinite primes in algebraic extension of rationals.

4.1.3 How infinite rationals correspond to quantum states and space-time surfaces?

The most promising answer to the question how infinite rationals correspond to space-time surfaces is discussed in detail in the next section. Here it is enough to give only the basic idea.

1. In zero energy ontology hyper-octonionic units (in real sense) defined by ratios of infinite integers have an nterpretation as representations for pairs of positive and negative energy states. Suppose that the quantum number combinations characterizing positive and negative energy quantum states are representable as superpositions of real units defined by ratios of infinite integers at each point of the space-time surface. If this is true, the quantum classical correspondence coded by the measurement interaction term of the Kähler-Dirac action maps the quantum numbers also to space-time geometry and implies a correspondence between infinite rationals and space-time surfaces.

2. The space-time surface associated with the infinite rational is in general not a union of the space-time surfaces associated with the primes composing the integers defining the rational. There the classical description of interactions emerges automatically. The description of classical states in terms of infinite integers would be analogous to the description of many particle states as finite integers in arithmetic quantum field theory. This mapping could in principle make sense both in real and p-adic sectors of WCW .

The finite primes which correspond to particles of an arithmetic quantum field theory present in Fock state, correspond to the space-time sheets of finite size serving as the building blocks of the space-time sheet characterized by infinite prime.

4.1.4 What is the interpretation of the higher level infinite primes?

Infinite hierarchy of infinite primes codes for a hierarchy of Fock states such that many-particle Fock states of a given level serve as elementary particles at next level. The unavoidable conclusion is that higher levels represent totally new physics not described by the standard quantization procedures. In particular, the assignment of fermion/boson property to arbitrarily large system would be in some sense exact. Topologically these higher level particles could correspond to space-time sheets containing many-particle states and behaving as higher level elementary particles.

This view suggests that the generating quantum numbers are present already at the lowest level and somehow coded by the hyper-octonionic primes taking the role of momentum quantum number they have in arithmetic quantum field theories. The task is to understand whether and how hyper-octonionic primes can code for quantum numbers predicted by quantum TGD.

The quantum numbers coding higher level states are collections of quantum numbers of lower level states. At geometric level the replacement of the coefficients of polynomials with rational functions is the equivalent of replacing single particle states with new single particle states consisting of many-particle states.

4.2 Infinite Primes, The Structure Of Many-Sheeted Space-Time, And The Notion Of Finite Measurement Resolution

The mapping of infinite primes to space-time surfaces codes the structure of infinite prime to the structure of space-time surface in a rather non-implicit manner, and the question arises about the concrete correspondence between the structure of infinite prime and topological structure of the space-time surface. It turns out that the notion of finite measurement resolution is the key concept: infinite prime characterizes angle measurement resolution. This gives a direct connection with the p-adicization program relying also on angle measurement resolution as well as a connection with the hierarchy of Planck constants. Finite measurement resolution relates also closely to the inclusions of hyper-finite factors central for TGD inspired quantum measurement theory so that the characterization of the finite measurement resolution, which has been the ugly ducling of theoretical physics transforms to a beautiful swan.

4.2.1 The first intuitions

The concrete prediction of the general vision is that the hierarchy of infinite primes should somehow correspond to the hierarchy of space-time sheets or partonic 2-surfaces if one accepts the effective 2-dimensionality. The challenge is to find space-time counterparts for infinite primes at the lowest level of the hierarchy.

One could hope that the Fock space structure of infinite prime would have a more concrete correspondence with the structure of the many-sheeted space-time. One might that the space-time sheets labeled by primes p would directly correspond to the primes appearing in the definition of infinite prime. This expectation seems to be too simplistic.

1. What seems to be a safe guess is that the simplest infinite primes at the lowest level of the hierarchy should correspond to elementary particles. If inverses of infinite primes correspond to negative energy space-time sheets, this would explain why negative energy particles are not encountered in elementary particle physics.

2. More complex infinite primes at the lowest level of the hierarchy could be interpreted in terms of structures formed by connecting these structures by flux tubes to get space-time correlates of bound states. Even simplest infinite primes must correspond to bound state structures if the condition that the corresponding polynomial has real-rational coefficients is taken seriously.

Infinite primes at the lowest level of hierarchy correspond to several finite primes rather than single finite prime. The number of finite primes is however finite.

1. A possible interpretation for multi-p property is in terms of multi-p p-adic fractality prevailing in the interior of space-time surface. The effective p-adic topology of these space-time sheets would depend on length scale. In the longest scale the topology would correspond to p_n, in some shorter length scale there would be smaller structures with $p_{n-1} < p_n$-adic topology, and so on.... A good metaphor would be a wave containing ripples, which in turn would contain still smaller ripples. The multi-p p-adic fractality would be assigned with the 4-D space-time sheets associated with elementary particles. The concrete realization of multi-p p-adicity would be in terms of infinite integers coming as power series $\sum x_n N^n$ and having interpretation as p-adic numbers for any prime dividing N.

2. Effective 2-dimensionality would suggest that the individual p-adic topologies could be assigned with the 2-dimensional partonic surfaces. Thus infinite prime would characterize at the lowest level space-time sheet and corresponding partonic 2-surfaces. There are however reasons to think that even single partonic 2-surface corresponds to a multi-p p-adic topology.

4.2.2 Do infinite primes code for the finite measurement resolution?

The above describe heuristic picture is not yet satisfactory. In order to proceed, it is good to ask what determines the finite prime or set of them associated with a given partonic 2-surface. It is good to recall first the recent view about the p-adicization program relying crucially on the notion of finite measurement resolution.

1. The vision about p-adicization characterizes finite measurement resolution for angle measurement in the most general case as $\Delta\phi = 2\pi M/N$, where M and N are positive integers having no common factors. The powers of the phases $exp(i2\pi M/N)$ define identical Fourier

basis irrespective of the value of M and measurement resolution does not depend on on the value of M. Situation is different if one allows only the powers $exp(i2\pi kM/N)$ for which $kM < N$ holds true: in the latter case the measurement resolutions with different values of M correspond to different numbers of Fourier components. If one regards N as an ordinary integer, one must have $N = p^n$ by the p-adic continuity requirement.

2. One can also interpret N as a p-adic integer. For $N = p^n M$, where M is not divisible by p, one can express $1/M$ as a p-adic integer $1/M = \sum_{k\geq 0} M_k p^k$, which is infinite as a real integer but effectively reduces to a finite integer $K(p) = \sum_{k=0}^{N-1} M_k p^k$. As a root of unity the entire phase $exp(i2\pi M/N)$ is equivalent with $exp(i2\pi R/p^n)$, $R = K(p)M \mod p^n$. The phase would non-trivial only for p-adic primes appearing as factors in N. The corresponding measurement resolution would be $\Delta\phi = R2\pi/N$ if modular arithetics is used to define the measurement resolution. This works at the first level of the hierarchy but not at higher levels. The alternative manner to assign a finite measurement resolution to M/N for given p is as $\Delta\phi = 2\pi|N/M|_p = 2\pi/p^n$. In this case the small fermionic part of the infinite prime would fix the measurement resolution. The argument below shows that only this option works also at the higher levels of hierarchy and is therefore more plausible.

3. p-Adicization conditions in their strong form require that the notion of integration based on harmonic analysis in symmetric spaces makes sense even at the level of partonic 2-surfaces. These conditions are satisfied if the partonic 2-surfaces in a given measurement resolution can be regarded as algebraic continuations of discrete surfaces whose points belong to the discrete variant of the $\delta M^4_\pm \times CP_2$. This condition is extremely powerful since it effectively allows to code the geometry of partonic 2-surfaces by the geometry of finite sub-manifold geometries for a given measurement resolution. This condition assigns the integer N to a given partonic surface and all primes appearing as factors of N define possible effective p-adic topologies assignable to the partonic 2-surface.

How infinite primes could then code for the finite measurement resolution? Can one identify the measurement resolution for $M/N = M/(Rp^n)$ as $\Delta\phi = ((M/R)\ mod\ p^n) \times 2\pi/p^n$ or as $\Delta\phi = 2\pi/p^n$? The following argument allows only the latter option.

1. Suppose that p-adic topology makes sense also for infinite primes and that state function reduction selects power of infinite prime P from the product of lower level infinite primes defining the integer N in M/N. Suppose that the rational defined by infinite integer defines measurement resolution also at the higher levels of the hierarchy.

2. The infinite primes at the first level of hierarchy representing Fock states are in one-one correspondence with finite rationals M/N for which integers M and N can be chosen to characterize the infinite bosonic part and finite fermionic part of the infinite prime. This correspondence makes sense also at higher levels of the hierarchy but M and N are infinite integers. Also other option obtained by exchanging "bosonic" and "fermionic" but later it will be found that only the first identification makes sense.

3. The first guess is that the rational M/N characterizing the infinite prime characterizes the measurement resolution for angles and therefore partially classifies also the finite sub-manifold geometry assignable to the partonic 2-surface. One should define what $M/N = ((M/R)\ mod\ P^n) \times P^{-n}$ is for infinite primes. This would require expression of M/R in modular arithmetics modulo P^n. This does not make sense.

4. For the second option the measurement resolution defined as $\Delta\phi = 2\pi|N/M|_P = 2\pi/P^n$ makes sense. The Fourier basis obtained in this manner would be infinite but all states $exp(ik/P^n)$ would correspond in real sense to real unity unless one allows k to be infinite P-adic integer smaller than P^n and thus expressible as $k = \sum_{m<n} k_m P^m$, where k_m are infinite integers smaller than P. In real sense one obtains all roots $exp(iq2\pi)$ of unity with $q < 1$ rational. For instance, for $n = 1$ one can have $0 < k/P < 1$ for a suitably chosen infinite prime k. Thus one would have essentially continuum theory at higher levels of the hierarchy. The purely fermionic part N of the infinite prime would code for both the number of Fourier components in discretization for each power of prime involved and the ratio characterize the angle resolution.

The proposed relation between infinite prime and finite measurement resolution implies very strong number theoretic selection rules on the reaction vertices.

1. The point is that the vertices of generalized Feynman diagrams correspond to partonic 2-surfaces at which the ends of light-like 3-surfaces describing the orbits of partonic 2-surfaces join together. Suppose that the partonic 2-surfaces appearing a both ends of the propagator lines correspond to same rational as finite sub-manifold geometries. If so, then for a given p-adic effective topology the integers assignable to all lines entering the vertex must contain this p-adic prime as a factor. Particles would correspond to integers and only the particles having common prime factors could appear in the same vertex.

2. In fact, already the work with modelling dark matter [K5] led to ask whether particle could be characterized by a collection of p-adic primes to which one can assign weak, color, em, gravitational interactions, and possibly also other interactions. It also seemed natural to assume that that only the space-time sheets containing common primes in this collection can interact. This inspired the notions of relative and partial darkness. An entire hierarchy of weak and color physics such that weak bosons and gluons of given physics are characterized by a given p-adic prime p and also the fermions of this physics contain space-time sheet characterized by same p-adic prime, say M_{89} as in case of weak interactions. In this picture the decay widths of weak bosons do not pose limitations on the number of light particles if weak interactions for them are characterized by p-adic prime $p \neq M_{89}$. Same applies to color interactions.

The possibility of multi-p p-adicity raises the question about how to fix the p-adic prime characterizing the mass of the particle. The mass scale of the contribution of a given throat to the mass squared is given by $p^{-n/2}$, where $T = 1/n$ corresponds to the p-adic temperature of throat. Hence the dominating contribution to the mass squared corresponds to the smallest prime power p^n associated with the throats of the particle. This works if the integers characterizing other particles than graviton are divisible by the gravitonic p-adic prime or a product of p-adic primes assignable to graviton. If the smallest power p^n assignable to the graviton is large enough, the mass of graviton is consistent with the empirical bounds on it. The same consideration applies in the case of photons. Recall that the number theoretically very natural condition that in zero energy ontology the number of generalized Feynman graphs contributing to a given process is finite is satisfied if all particles have a non-vanishing but arbitrarily small p-adic thermal mass [K14].

4.2.3 Interpretational problem

The identification of infinite prime as a characterizer of finite measurement resolution looks nice but there is an interpretational problem.

1. The model characterizing the quantum numbers of WCW spinor fields to be discussed in the next section involves a pair of infinite primes P_+ and P_- corresponding to the two vacuum primes $X \pm 1$. Do they correspond to two different measurement resolutions perhaps assignable to CD and CP_2 degrees of freedom?

2. Different measurement resolutions in CD and CP_2 degrees of freedom need not be not a problem as long as one considers only the discrete variants of symmetric spaces involved. What might be a problem is that in the general case the p-adic primes associated with CD and CP_2 degrees of freedom would not be same unless the integers N_+ and N_- are assumed to have have same prime factors (they indeed do if $p^0 = 1$ is formally counted as prime power factors).

3. The idea of assigning different p-adic effective topologies to CD and CP_2 does not look attractive. Both CD and CP_2 and thus also partonic 2-surface could however possess simultaneously both p-adic effective topologies. This kind of option might make sense since the integers representable as infinite powers series of integer N can be regarded as p-adic integers for all prime factors of N. As a matter fact, this kind of multi-p p-adicity could make sense also for the partonic 2-surfaces characterized by a measurement resolution $\Delta\phi = 2\pi M/N$. One would have what might be interpreted as N_+N_--adicity.

4. It will be found that quantum measurement means also the measurement of the p-adic prime selecting same p-adic prime from N_+ and N_-. If $N_\pm$ is divisible only by $p^0 = 1$, the corresponding angle measurement resolution is trivial. From the point of view of consciousness state function reduction selects also the p-adic prime characterizing the cognitive representation which is very natural since quantum superpositions of different p-adic topologies are not natural physically.

4.3 How The Hierarchy Of Planck Constants Could Relate To Infinite Primes And P-Adic Hierarchy?

Besides the hierarchy of space-time sheets, TGD predicts, or at least suggests, several hierarchies such as the hierarchy of infinite primes, the hierarchy of Jones inclusions identifiable in terms of finite measurement resolution [K13], the dark matter hierarchy characterized by increasing values of $\hbar$ [K4, K3], the hierarchy of extensions of given p-adic number field, and the hierarchy of selves and quantum jumps with increasing duration with respect to geometric time. There are good reasons to expect that these hierarchies are closely related. Number theoretical considerations give hopes about developing a more quantitative vision about the relationship between these hierarchies, in particular between the hierarchy of infinite primes, p-adic length scale hierarchy, and the hierarchy if Planck constants.

If infinite primes code for the hierarchy of measurement resolutions, the correlations between the p-adic hierarchy and the hierarchy of Planck constants indeed suggest themselves and allow also to select between two interpretations for the fact that two infinite primes N_+ and N_- are needed to characterize elementary particles (see the next section).

Recall that the hierarchy of Planck constants in the most general situation corresponds to a replacement M^4 and CP_2 factors of the imbedding space with singular coverings and factor spaces. The condition that Planck constant is integer valued allows only singular coverings characterized by two integers n_a *resp.* n_b assignable to CD *resp.* CP_2. This condition also guarantees that a given value of Planck constant corresponds to only a finite number of pages of the “Big Book” and therefore looks rather attractive mathematically. This option also forces evolution as a dispersion to the pages of the books characterized by increasing values of Planck constant.

Concerning the correspondence between the hierarchy of Planck constants and p-adic length scale hierarchy there seems to be only single working option. The following assumptions make precise the relationship between finite measurement resolution, infinite primes and hierarchy of Planck constants.

1. Measurement resolution CD *resp.* CP_2 degrees of freedom is assumed to correspond to the rational M_+/N_+ *resp.* M_-/N_-. $N_\pm$ is identified as the integer assigned to the fermionic part of the infinite integer..

2. One must always fix the consideration to a fixed p-adic prime. This process could be regarded as analogous to fixing the quantization axes and p would also characterize the p-adic cognitive space-time sheets involved. The p-adic prime is therefore same for CD and CP_2 degrees of freedom as required by internal consistency.

3. The relationship to the hierarchy of Planck constants is fixed by the identifications $n_a = n_+(p)$ and $n_b = n_-(p)$ so that the number of sheets of the covering equals to the number of bosons in the fermionic mode p of the quantum state defined by infinite prime.

4. A physically attractive hypothesis is that number theoretical bosons *resp.* fermions correspond to WCW orbital *resp.* spin degrees of freedom. The first ones correspond to the symplectic algebra of WCW and the latter one to purely fermionic degrees of freedom.

Consider now the basic consequences of these assumptions from the point of view of physics and cognition.

1. Finite measurement resolution reduces for a given value of p to

$$\Delta\phi = \frac{2\pi}{p^{n_\pm(p)+1}} = \frac{2\pi}{p^{n_{a/b}}} \quad ,$$

where $n_\pm(p) = n_{a/b} - 1$ is the number of bosons in the mode p in the fermionic part of the state. The number theoretical fermions and bosons and also their probably existing physical counterparts are necessary for a non-trivial angle measurement resolution. The value of Planck constant given by

$$\frac{\hbar}{\hbar_0} = n_a n_b = (n_+(p) + 1) \times (n_-(p) + 1)$$

tells the total number of bosons added to the fermionic mode p assigned to the infinite prime.

2. The presence of $\hbar > \hbar_0$ partonic 2-surfaces is absolutely essential for a Universe able to measure its own state. This is in accordance with the interpretation of hierarchy of Planck constants in TGD inspired theory of consciousness. One can also say that $\hbar = 0$ sector does not allow cognition at all since $N_\pm = 1$ holds true. For given p $\hbar = n_a n_b = 0$ means that given fermionic prime corresponds to a fermion in the Dirac sea meaning $n_\pm(p) = -1$. Kicking out of fermions from Direac sea makes possible cognition. For purely bosonic vacuum primes one has $\hbar = 0$ meaning trivial measurement resolution so that the physics is purely classical and would correspond to the purely bosonic sector of the quantum TGD.

3. For $\hbar = \hbar_0$ the number of bosons in the fermionic state vanishes and the general expression for the measurement resolution reduces to $\Delta\phi = 2\pi/p$. When one adds $n_\pm(p)$ bosons to the fermionic part of the infinite prime, the measurement resolution increases from $\Delta\phi = 2\pi/p$ to $\Delta\phi = 2\pi/p^{n_\pm(p)+1}$. Adding a sheet to the covering means addition of a number theoretic boson to the fermionic part of infinite prime. The presence of both number theoretic bosons and fermions with the values of p-adic prime $p_1 \neq p$ does not affect the measurement resolution $\Delta\phi = 2\pi/p^n$ for a given prime p.

4. The resolutions in CD and CP_2 degrees of freedom correspond to the same value of the p-adic prime p so that one has dicretizations based on $\Delta\phi = 2\pi/p^{n_a}$ in CD degrees of freedom and $\Delta\phi = 2\pi/p^{n_b}$ in CP_2 degrees of freedom. The finite sub-manifold geometries make sense in this case and since the effective p-adic topology is same, the continuation to continuous p-adic partonic 2-surface is possible.

p-Adic thermodynamics involves the p-adic temperature $T = 1/n$ as basic parameter and the p-adic mass scale of the particle comes as $p^{-(n+1)/2}$. The natural question is whether one could assume the relation $T_\pm = 1/(n_\pm(p) + 1)$ between p-adic temperature and infinite prime and thus the relations $T_a = 1/n_a(p)$ and $T_b = 1/n_b(p)$. This identification is not consistent with the recent physical interpretation of the p-adic thermodynamics nor with the view about dark matter hierarchy and must be given up.

1. The minimal non-trivial measurement resolution with $n_i = 1$ and $\hbar = \hbar_0$ corresponds to the p-adic temperature $T_i = 1$. p-Adic mass calculations indeed predict $T = 1$ for fermions for $\hbar = \hbar_0$. In the case of gauge bosons $T \geq 2$ is favored so that gauge bosons would be dark. This would require that gauge bosons propagate along dark pages of the Big Book and become "visible" before entering to the interaction vertex.

2. p-Adic thermodynamics also assumes same p-adic temperature in CD and CP_2 degrees of freedom but the proposed identification allows also different temperatures. In principle the separation of the super-conformal degrees of freedom of CD and CP_2 might allow different p-adic temperatures. This would assign to different p-adic mass scales to the particles and the larger mass scale should give the dominant contribution.

3. For dark particles the p-adic mass scale would be by a factor $1/\sqrt{p}^{n_i(p)-1}$ lower than for ordinary particles. This is in conflict with the assumption that the mass of the particle does not depend on $\hbar$. This prediction would kill completely the recent vision about the dark matter.

5 How Infinite Primes Could Correspond To Quantum States And Space-time Surfaces?

The hierarchy of infinite primes is in one-one correspondence with a hierarchy of second quantizations of an arithmetic quantum field theory. The additive quantum number in question is energy like quantity for ordinary primes and given by the logarithm of prime whereas p-adic length scale hypothesis suggests that the conserved quantity is proportional to the inverse of prime or its square root. For infinite primes at the first level of hierarchy these quantum numbers label single particles states having interpretation as ordinary elementary particles. For octonionic and hyper-octonionic primes the quantum number is analogous to a momentum with 8 components. The question is whether these number theoretic quantum numbers could have interpretation as genuine quantum numbers. Quantum classical correspondence raises another question. Is it possible to label space-time surfaces by infinite primes? Could this correspondence be even one-to-one?

I have considered these questions already more than decade ago. The discussion at that time was necessarily highly speculative and just a mathematical exercise. After that time however a lot of progress has taken place in quantum TGD and it is highly interaction to see what comes out from the interaction of the notion of infinite prime with the notions of zero energy ontology and generalized imbedding space, and with the recent vision about how Chern-Simons Dirac term in the Kähler-Dirac action allows to code information about quantum numbers to the space-time geometry. The possibility of this coding allows to simplify the discussion dramatically. If one could map infinite hyper-octonionic or hyper-quaternionic primes to quantum numbers of the standard model naturally, then the their map of to the geometry of space-time surfaces would realize the coding of space-time surfaces by infinite primes (and more generally by integers and rationals). Also a detailed realization of number theoretic Brahman=Atman identity would emerge as an outcome.

5.1 A Brief Summary About Various Moduli Spaces And Their Symmetries

It is good to sum up the number theoretic symmetries before trying to construct an overall view about the situation. Several kinds of number theoretical symmetry groups are involved corresponding to symmetries in the moduli spaces of hyper-octonionic and hyper-quaternionic structures, symmetries mapping hyper-octonionic primes to hyper-octonionic primes, and translations acting in the space of causal diamonds (CDs) and shifting. The moduli space for CDs labeled by pairs of its tips that its pairs of points of $M^4 \times CP_2$ is also in important role.

1. The basic idea is that color $SU(3) \subset G_2$ acts as automorphisms of hyper-octonion structure with a preferred imaginary unit. $SO(7,1)$ acts as symmetries in the moduli space of hyper-octonion structures. Associativity implies symmetry breaking so that only hyper-quaternionic structures are considered and $SO(3,1) \times SO(4)$ acts as symmetries of the moduli space for hyper-quaternionic structures.

2. CP_2 parameterizes the moduli space of hyper-quaternionic structures induced from a given hyper-octonionic structure with preferred imaginary unit.

3. Color group $SU(3)$ is the analog of Galois group for the extension of reals to octonions and has a natural action on the decompositions of rational infinite primes to hyper-octonionic infinite primes. For given hyper-octonionic prime one can identify a subgroup of SU(3) generating a finite set of hyper-octonionic primes for it at sphere S^7. This suggests wave function at the orbit of given hyper-octonionic prime in turn generalizing to wave functions in the space of infinite primes.

4. Four-momenta correspond to translational degrees of freedom associated with the preferred points of M^4 coded by the infinite rational (tip of the light-cone). Color quantum numbers in cm degrees of freedom can be assigned to the CP_2 projection of the preferred point of H. As a matter fact, the definition of hyper-octonionic structure involves the choice of origin of M^8 giving rise to the preferred point of H.

These symmetries deserve a more detailed discussion.

1. The choice of global hyper-octonionic coordinate is dictated only modulo a transformation of $SO(1,7)$ acting as isometries of hyper-octonionic norm and as transformations in moduli space of hyper-octonion structures. $SO(7)$ respects the choice of the real unit. $SO(1,3) \times SO(4)$ acts in the moduli space of global hyper-quaternionic structures identified as sub-structures of hyper-octonionic structure. The choice of global hyper-octonionic structures involves also a choice of origin implying preferred point of H. The M^4 projection of this point corresponds to the tip of CD. Since the integers representing physical states must be hyper-quaternionic by associativity conditions, the symmetry breaking ("number theoretic compactification") to $SO(1,3) \times SO(4)$ occurs very naturally. This group acts as spinor rotations in H picture and as isometries in M^8 picture. The choice of both tips of CD reduces $SO(1,3)$ to $SO(3)$.

2. $SO(1,7)$ allows 3 different 8-dimensional representations (8_v, 8_s, and $\overline{8}_s$). All these representations must decompose under $SU(3)$ as $1+1+3+\overline{3}$ as little exercise with $SO(8)$ triality demonstrates. Under $SO(6) \cong SU(4)$ the decompositions are $1+1+6$ and $4+\overline{4}$ for 8_v and 8_s and its conjugate. Both hyper-octonion spinors and gamma matrices are identified as hyper-octonion units rather than as matrices. It would be natural to assign to bosonic M^8 primes 8_v and to fermionic M^8 primes 8_s and $\overline{8}_s$. One can distinguish between $8_v, 8_s$ and $\overline{8}_s$ for hyper-octonionic units only if one considers the full $SO(1,3) \times SO(4)$ action in the moduli space of hyper-octonionic structures.

3. G_2 acts as automorphisms on octonionic imaginary units and $SU(3)$ respects the choice of preferred imaginary unit meaning a choice of preferred hyper-complex plane $M^4 \subset M^4$. Associativity requires a reduction to hyper-quaternionic primes and implies color confinement in number theoretical and as it turns also in physical sense. For hyper-quaternionic primes the automorphisms restrict to $SO(3)$ which has right/left action of fermionic hyper-quaternionic primes and adjoint action on bosonic hyper-quaternionc primes. The choice of hyper-quaternionic structure is global as opposed to the local choice of hyper-quaternionic tangent space of space-time surface assigning to a point of $HQ \subset HO$ a point of CP_2. $U(2) \subset SU(3)$ leaves invariant given hyper-quaternionic structure which are thus parameterized by CP_2. Color partial waves can be interpreted as partial waves in this moduli space.

5.2 Associativity And Commutativity Or Only Their Quantum Variants?

Associativity and commutativity conditions are absolutely essential notions in quantum TGD and also in the mapping of infinite primes to the space-time sheets. Hyper-quaternionicity formulated in terms of the Kähler-Dirac gamma matrices defined by Kähler action fixes classical space-time dynamics and a very beautiful algebra formulation of quantum TGD in terms of hyper-octonionic local Clifford algebra of imbedding space emerges. There is no need for the use of hyper-octonion real analytic maps although one cannot exclude the possibility that they might be involved with the construction of hyper-quaternionic space-time surfaces.

Associativity implies hyper-quaternionicity and commutativity requirement in turn leads to complex rational infinite primes. Since one can decompose complex rational primes to hyper-quaternionic and even hyper-octonionic primes, one might hope that this could allow to represent states which consist of colored constituents. This representations has however the flavor of a formal trick and the considerations related to concrete representations of infinite primes suggest that the rationality of infinite primes might be a too restrictive condition.

A more radical possibility is that physical states are only quantum associative and commutative. In case of associativity this means that they are obtained as quantum superpositions in the space of real units over all possible associations performed for a given product of hyper-octonion primes (for instance, $|A(BC)\rangle + |(AB)C\rangle$). These states would be associative in quantum sense but would not reduce to hyper-quaternionic primes. Also the notion of quantum commutativity makes sense. The fact that mesons are quantum superpositions of quark-antiquark pairs which each corresponds to different pair of hyper-quaternionic primes and are thus not representable classically, suggests that one can require only quantum associativity and quantum commutativity.

5.3 How Space-Time Geometry Could Be Coded By Infinite Primes?

Second key question is whether space-time geometry could be characterized in terms of infinite primes (and integers and rationals in the most general case) and how this is achieved. The question is how the quantum states consisting of fundamental fermions serving as building bricks of elementary particles could be coded by infinite quaternionic integeres to which one can assign ordinary finite quaternionic primes.

The basic idea is roughly that at the first level of the hierarchy the finite primes appearing as building blocks of infinite prime correspond to structures formed by pairs or wormhole contacts assigned with elementary particles.

1. The partonic orbits defined by wormhole throats could be characterized by finite primes specifying the preferred p-adic topology assignable to the p-adic "cognitive representation" of the throat.

2. One could assign hyper-quaternionic integer to the real particle as its four-momentum. In this case the mass shell condition would fix the hyper-quaternionic integer to a high extent. All discrete Lorentz boosts of the particle state taking hyper-quaternionic integers to hyper-quaternionic integers would correspond to the same p-adic integer (prime) defined by the length of the Lorentz boosted hyper-quaternionic integer. The p-adic prime characterizing virtual particle would be one of the primes appearing in the factorization of this integer to a product of powers of prime, most naturally the one whose power is largest.

   Note that p-adic length scale hypothesis suggests that the p-adic primes near powers of two are favored for on mass shell particles and perhaps also for the virtual particles.

3. For fundamental fermions associated with boundaries of string world sheets and appearing as building bricks of particles the masses would vanish on mass shell so that the hyper-quaternionic integer would in this case have vanishing norm.

   The virtual four-momentum assigned to a virtual fermion line as a generalized eigenvalue of Chern-Simons Dirac operator would correspond to hyper-quaternionic integer. In this case p-adic prime would be defined as for physical particles and would depend on the mass of the virtual particle. If the integration over virtual momenta by residue calculus effectively leads to an integral over on mass shell massless virtual momenta with non-physical spinor helicities then also virtual fundamental fermions would correspond to zero norm hyper-quaternionic integers.

4. The correlation between particle's four-momentum and the p-adic prime characterizing corresponding cognitive representation would be in accordance with quantum classical correspondence.

5. The hyperquaternionic primes appearing as largest factors in the factorizion of hyper-quaternionic integers assignable with physical particles could be interpreted as building bricks of an infinite hyperquaternionic prime characterizing the many-particle state and at least the boundaries of string world sheets. The idea that p-adic space-time surfaces defined "cognitive representations" as p-adic chart maps of real space-time surfaces and vice versa (as the TGD based definition of p-adic manifolds assumes) suggests that the p-adic primes in question characterize also space-time regions rather than only the boundaries of string world sheets.

A couple of comments about this speculation are in order.

1. ZEO implies a hierarchy of CDs within CDs and this hierarchy as well as the hierarchy of space-time sheets corresponds naturally to the hierarchy of infinite primes. One can assign standard model quantum numbers to various partonic 2-surfaces with positive and negative energy parts of the quantum state assignable to the light-like boundaries of CD. Also infinite integers and rationals are possible and the inverses of infinite primes would

naturally correspond to elementary particles with negative energy. The condition that zero energy state has vanishing net quantum numbers implies that the ratio of infinite integers assignable to zero energy state equals to real unit in real sense and has has vanishing total quantum numbers.

2. Neither quantum numbers nor infinite primes coding them cannot characterize the partonic 2-surface itself completely since they say nothing about the deformation of the space-time surface but only about labels characterizing the WCW spinor field. Also the topology of partonic 2-surface fails to be coded. Quantum classical correspondence however suggests that this correspondence could be possible in a weaker sense. In the Gaussian approximation for functional integral over the world of classical worlds space-time surface and thus the collection of partonic 2-surfaces is effectively replaced with the one corresponding to the maximum of Kähler function, and in this sense one-one correspondence is possible unless the situation is non-perturbative. In this case the physics implied by the hierarchy of Planck constants could however guarantee uniqueness.

   One of the basic ideas behind the identification of the dark matter as phases with non-standard value of Planck constant is that when perturbative description of the system fails, a phase transition increasing the value of Planck constant takes place and makes perturbative description possible. Geometrically this phase transition means a leakage to another sector of the imbedding space realized as a book like structure with pages partially labeled by the values of Planck constant. Anyonic phases and fractionization of quantum numbers is one possible outcome of this phase transition. An interesting question is what the fractionization of the quantum numbers means number theoretically.

6 Infinite Primes And Mathematical Consciousness

The mathematics of infinity relates naturally with the mystery of consciousness and religious and mystic experience. In particular, mathematical cognition might have as a space-time correlate the infinitely structured space-time points implied by the introduction of infinite-dimensional space of real units defined by infinite (hyper-)octonionic rationals having unit norm in the real sense. I hope that the reader takes this section as a noble attempt to get a glimpse about unknown rather than final conclusions.

6.1 Algebraic Brahman=Atman Identity

The proposed view about cognition and intentionality emerges from the notion of infinite primes, which was actually the first genuinely new mathematical idea inspired by TGD inspired consciousness theorizing. Infinite primes, integers, and rationals have a precise number theoretic anatomy. For instance, the simplest infinite primes correspond to the numbers $P_{\pm} = X \pm 1$, where $X = \prod_k p_k$ is the product of all finite primes. Indeed, $P_{\pm} \ mod \ p = 1$ holds true for all finite primes. The construction of infinite primes at the first level of the hierarchy is structurally analogous to the quantization of super-symmetric arithmetic quantum field theory with finite primes playing the role of momenta associated with fermions and bosons. Also the counterparts of bound states emerge. This process can be iterated at the second level the product of infinite primes constructed at the first level replaces X and so on.

The structural similarity with repeatedly second quantized quantum field theory strongly suggests that physics might in some sense reduce to a number theory for infinite rationals M/N and that second quantization could be followed by further quantizations. As a matter fact, the hierarchy of space-time sheets could realize this endless second quantization geometrically and have also a direct connection with the hierarchy of logics labeled by their order. This could have rather breathtaking implications.

1. One is forced to ask whether this hierarchy corresponds to a hierarchy of realities for which level below corresponds in a literal sense infinitesimals and the level next above to infinity.

2. Second implication is that there is an infinite number of infinite rationals behaving like real units ($M/N \equiv 1$ in real sense) so that space-time points could have infinitely rich

number theoretical anatomy not detectable at the level of real physics. Infinite integers would correspond to positive energy many particle states and their inverses (infinitesimals with number theoretic structure) to negative energy many particle states and $M/N \equiv 1$ would be a counterpart for zero energy ontology to which oneness and emptiness are assigned in mysticism.

3. Single space-time point, which is usually regarded as the most primitive and completely irreducible structure of mathematics, would take the role of Platonia of mathematical ideas being able to represent in its number theoretical structure even the quantum state of entire Universe. Algebraic Brahman=Atman identity and algebraic holography would be realized in a rather literal sense.

6.1.1 Number theoretic anatomy of space-time point

This number theoretical anatomy should relate to mathematical consciousness in some manner. For instance, one can ask whether it makes sense to speak about quantum jumps changing the number theoretical anatomy of space-time points and whether these quantum jumps give rise to mathematical ideas. In fact, the identifications of Platonia as spinor fields in WCW on one hand and as the set number theoretical anatomies of point of imbedding space force the conclusion that WCW spinor fields (recall also the identification as correlates for logical mind) can be realized in terms of the space for number theoretic anatomies of imbedding space points. Therefore quantum jumps would be correspond to changes in the anatomy of the space-time points. Or more precisely, to the changes of the WCW spinor fields regarded as wave functions in the set of imbedding space points which are equivalent in real sense. Imbedding space would be experiencing genuine number theoretical evolution. The whole physics would reduce to the anatomy of numbers. All mathematical notions which are more than mere human inventions would be imbeddable to the Platonia realized as the number theoretical anatomies of single imbedding space point.

To realize this picture would require that WCW spinor fields and perhaps even WCW allow a mapping to the number theoretic anatomies of space-time point. In finite-dimension Euclidian spaces momentum space labelling plane wavs is dual to the space. One could hope that also now the "orbital" quantum numbers of WCW spinor fields could code for WCW in given measurement resolution. The construction of the previous sections realize the mapping of the quantum states defined by WCW spinors fields assignable to given CD to wave function in the space of hyper-octonionic units. These wave functions can be also regarded as linear combinations of these units if the coefficients are complex numbers formed using the commuting imaginary unit of complexified octonions so that that the Hilbert space like structure in question would have purely number theoretic meaning. The rationals defined by infinite primes characterize also measurement resolution and classify the the finite sub-manifold geometries associated with partonic two-surfaces. At higher levels one has rationals defined by ratios of infinite integers and one can ask whether this interpretation generalizes.

Note that one must distinguish between two kinds of hyper-octonionic units.

1. Already in the case of complex numbers one has rational complex units defined in terms of Pythagorean triangle and their products generate infinite dimensional space. The hyper-octonionic units defined as ratios U of infinite integers and suggested to provide a representation of WCW spinor fields correspond to these. The powers U^m define roots of unity which can be regarded analogous to $exp(i2\pi x)$, where x is not rational but the exponent itself is complex rational.

2. Besides this there are roots of unity which are in general algebraic complex numbers. These roots of unit correspond to phases $exp(i2\pi M/N)$, where M/N is ratio of real infinite integers and i is the commuting hyper-octonionic imaginary unit. These real infinite integers can be assigned to hyper-octonionic integers by replacing everywhere finite hyper-octonionic primes with their norm which is ordinary prime. By the previous considerations only the phases $exp(i2\pi M/P^n)$ make sense p-adically for infinite primes P.

6.2 Leaving The World Of Finite Reals And Ending Up To The Ancient Greece

If strong number theoretic vision is accepted, all physical predictions of quantum TGD would be numbers in finite algebraic extensions of rationals at the first level of hierarchy. Just the numbers which ancient Greeks were able to construct by the technical means at use! This seems rather paradoxical but conforms also with the hypothesis that the dicrete algebraic intersections of real and p-adic 2-surfaces provide the fundamental cognitive representations.

The proposed construction for infinite primes gives a precise division of infinite primes to classes: the ratios of primes in given class span a *subset of rational numbers*. These classes give much more refined classification of infinities than infinite ordinals or alephs. They would correspond to separate phases in the evolution of consciousness identified as a sequence of quantum jumps defining sequence of primes $\rightarrow p_1 \rightarrow p_2$...... Infinite primes could mean a transition from space-time level to the level of function spaces. WCW is example of a space which can be parameterized by a space of functions locally.

The minimal assumption is that infinite primes reflect their presence only in the possibility to multiply the coordinates of imbedding space points by real units formed as ratios of infinite integers. The correspondence between polynomials and infinite primes gives hopes of mapping at least the reduced WCW consisting of the maxima of Kähler function to the anatomy of space-time point. Also WCW spinors and perhaps also the modes of WCW spinor fields would allow this kind of map.

One can consider also the possibility that infinite integers and rationals give rise to a hierarchy of imbedding spaces such that given level represents infinitesimals from the point of view of higher levels in hierarchy. Even "simultaneous" time evolutions of conscious experiences at different aleph levels with completely different time scales (to put it mildly) are possible since the time values around which the contents of conscious experience are possibly located, are determined by the quantum jump: also multi-snapshots containing snapshots also from different aleph levels are possible. Un-integrated conscious experiences with all values of p could be contained in given quantum jump: this would give rise to a hierarchy of conscious beings: the habitants above given level could be called Gods with full reason: those above us would probably call us just "epsilons" if ready to admit that we exist at all except in non-rigorous formulations of elementary calculus!

6.3 Infinite Primes And Mystic World View

The proposed interpretation deserves some additional comments from the point of consciousness theory.

1. An open problem is whether the finite integer S appearing in the infinite prime is product of only finite or possibly even infinite number of lower level primes at a given level of hierarchy. The proposed physical identification of S indeed allows S to be a product of infinitely many primes. One can allow also M and N appearing in the infinite and infinite part to be contain infinite number of factors. In this manner one obtains a hierarchy of infinite primes expressible in the form

$$\begin{array}{l} P = nY^{r_1} + mS \ , \quad r = 1,2,... \\ m = m_0 + P_{r_2}(Y) \ , \\ Y = \frac{X}{S} \ , \\ S = \prod_i P_i \ . \end{array}$$

   Note that this ansatz is in principle of the same general form as the original ansatz $P = nY + mS$. These primes correspond in physical analogy to states containing infinite number of particles.

   If one poses no restrictions on S this implies that that the cardinality for the set of infinite primes at first level would be $c = 2^{alef_0}$ ($alef_0$ is the cardinality of natural numbers). This is the cardinality for *all* subsets of natural numbers equal to the cardinality of reals. At the next level one obtains the cardinality 2^c for *all* subsets of reals, etc....

If S were always a product of *finite number of primes* and $k(p)$ would differ from zero for finite number of primes only, the cardinality of infinite primes would be $alef_0$ at each level. One could pose the condition that mS is infinitesimal as compared to nX/S. This would guarantee that the ratio of two infinite primes at the same level would be well defined and equal to n_1S_2/n_2S_1. On the other hand, the requirement that all rationals are obtained as ratios of infinite primes requires that no restrictions are posed on $k(p)$: in this case the cardinality coming from possible choices of $r = ms$ is the cardinality of reals at first level.

The possibility of primes for which also S is finite would mean that the algebra determined by the infinite primes must be generalized. For the primes representing states containing infinite number of bosons and/or fermions it would be be possible to tell how P_1P_2 and P_2P_1 differ and these primes would behave like elements of free algebra. As already found, this kind of free algebra would provide single space-time point with enormous algebraic representative power and analog of Brahman=Atman identity would result.

2. There is no physical subsystem-complement decomposition for the infinite primes of form $X \pm 1$ since fermionic degrees of freedom are not excited at all. Mystic could interpret it as a state of consciousness in which all separations vanish and there is no observer-observed distinction anymore. A state of pure awareness would be in question if bosonic and fermionic excitations represent the contents of consciousness! Since fermionic many particle states identifiable as Boolean statements about basic statements are identified as representation for reflective level of consciousness, $S = 1$ means that the reflective level of consciousness is absent: enlightenment as the end of thoughts according to mystics.

   The mystic experiences of oneness ($S = 1!$), of emptiness (the subset of primes defined by S is empty!) and of the absence of all separations (there is no subsystem-complement separation and hence no division between observer and observed) could be related to quantum jumps to this kind of sectors of the WCW . In super-symmetric interpretation $S = 1$ means that state contains no fermions.

3. There is entire hierarchy of selves corresponding to the hierarchy of infinite primes and the relationship between selves at different levels of the hierarchy is like the relationship between God and human being. Infinite primes at the lowest level would presumably represent elementary particles. This implies a hierarchy for moments of consciousness and it would be un-natural to exclude the existence of higher level "beings" (one might call them Angels, Gods, etc...).

6.4 Infinite Primes And Evolution

The original argument leading to the notion of infinite primes was simple. Generalized unitarity implies evolution as a gradual increase of the p-adic prime labeling the WCW sector D_p to which the localization associated with quantum jump occurs. Infinite p-adic primes are forced by the requirement that p-adic prime increases in a statistical sense and that the number of quantum jumps already occurred is infinite (assuming finite number of these quantum jumps and therefore the first quantum jump, one encounters the problem of deciding what was the first WCW spinor field).

Quantum classical correspondence requires that p-adic evolution of the space-time surface with respect to geometric time repeats in some sense the p-adic evolution by quantum jumps implied by the generalized unitarity [K7]. Infinite p-adic primes are in a well defined sense composites of the primes belonging to lower level of infinity and at the bottom of this de-compositional hierarchy are finite primes. This decomposition corresponds to the decomposition of the space-time surface into p-adic regions which in TGD inspired theory of consciousness correspond to selves. Therefore the increase of the composite primes at lower level of infinity induces the increase of the infinite p-adic prime. p-Adic prime can increase in two manners.

1. One can introduce the concept of the p-adic sub-evolution: the evolution of infinite prime P is induced by the sub-evolution of infinite primes belonging to a lower level of infinity being induced by.... being induced by the evolution at the level of finite primes. For instance, the increase of the cell size means increase of the p-adic prime characterizing it: neurons are

indeed very large and complicated cells whereas bacteria are small. Sub-evolution occurs both in subjective and geometric sense.

(a) For a given value of geometric time the p-adic prime of a given space-time sheet gradually increases in the evolution by quantum jumps: our geometric past evolves also!

(b) The p-adic prime characterizing space-time sheet also increases as the geometric time associated with the space-time sheet increases (say during morphogenesis).

The notion of sub-evolution is in accordance with the "Ontogeny recapitulates phylogeny" principle: the evolution of organism, now the entire Universe, contains the evolutions of the more primitive organisms as sub-evolutions.

2. Infinite prime increases also when entirely new finite primes emerge in the decomposition of an infinite prime to finite primes. This means that entirely new space-time sheets representing new structures emerge in quantum jumps. The creation of space-time sheets in quantum jumps could correspond to this process. By quantum classical correspondence this process corresponds at the space-time level to phase transitions giving rise to new material space-time sheets with more and more refined effective p-adic effective topology.

7 Does The Notion Of Infinite-P P-Adicity Make Sense?

In this section speculations related to infinite-P p-adicity are represented in the form of shy questions in order to not irritate too much the possible reader. The basic open question causing the tension is whether infinite primes relate only to the physics of cognition or whether they might allow to say something non-trivial about the physics of matter too.

The following list of questions is rather natural with the background provided by the p-adic physics.

1. Can one generalize the notion of p-adic norm and p-adic number field to include infinite primes? Could one define the counterpart of p-adic topology for literally infinite values of p? Does the topology R_P for infinite values of P approximate or is it equivalent with real topology as p-adic topology at the limit of infinite p is assumed to do (at least in the sense that p-adic variants of Diophantine equations at this limit correspond to ordinary Diophantine equations)? This is is possible is suggested by the fact that sheets of 3-surface are expected to have infinite size and thus to correspond to infinite p-adic length scale.

2. Canonical identification maps p-adic numbers of unit norm to real numbers in the range $[0, p]$. Does the canonical identification map the p-adic numbers R_P associated with infinite prime to reals? Could the number fields R_P provide alternative formulations/generalizations of the non-standard analysis based on the hyper-real numbers of Robinson [A10] ?

3. The notion of finite measurement resolution for angle variables given naturally as a hierarchy $2\pi/p^n$ of resolutions for a given p-adic prime defining aa hierarchy of algebraic extension of p-adic numbers is central in the attempts to formulate p-adic variants of quantum TGD and fuse them with real number based quantum TGD [K10] . If p is replaced with an infinite prime, the angular resolution becomes ideal and the roots of unity $exp(2\pi m/p^n)$ are replaced with real units unless also the integer m is replaced with an infinite integer M so that the ratio M/P^n is finite rational number. Could this approach be regarded as alternative for real number based notion of phase angle?

The consideration of infinite primes need not be a purely academic exercise: for infinite values of p p-adic perturbation series contains only two terms and this limit, when properly formulated, could give excellent approximation of the finite p theory for large p. Using infinite primes one might obtain the real theory in this approximation.

The question discussed in this section is whether the notion of p-adic number field makes sense makes sense for infinite primes and whether it might have some physical relevance. One can formally introduce power series in powers of any infinite prime P and the coefficients can be taken to belong to any ordinary number field. In the representation by polynomials P-adic power series correspond to Laurent series in powers of corresponding polynomial and are completely finite.

For straightforward generalization of the norm all powers of infinite-P prime have vanishing norm. The infinite-p p-adic norm of infinite-p p-adic integer would be given by its finite part so that in this sense positive powers of P would represent infinitesimals. For Laurent series this would mean that the lowest term would give the whole approximation in the real topology. For finite-primes one could however replace the norm as a power of p by a power of some other number. This would allow to have a finite norm also for P-adic primes. Since the simplest P-adic primes at the lowest level of hierarchy define naturally a rational one might consider the possibility of defining the norm of P as the inverse of this rational.

7.1 Does Infinite-P P-Adicity Reduce To Q-Adicity?

Any non-vanishing p-adic number is expressible as a product of power of p multiplied by a p-adic unit which can be infinite as a normal integer and has pinary expansion in powers of p:

$$x = p^n(x_0 + \sum_{k>0} x_k p^k) \ , \ x_k \in \{0,..,p-1\} \ , \ x_0 > 0 \ . \tag{7.1}$$

The p-adic norm of x is given by $N_p(x) = p^{-n}$. Each unit has p-adic inverse which for finite integers is always infinite as an ordinary integer.

To define infinite-P p-adic numbers one must generalize the pinary expansion to a infinite-P p-adic expansion of an infinite rational. In particular, one must identify what the statement "infinite integer modulo P" means when P is infinite prime, and what are the infinite integers N satisfying the condition $N < P$. Also one must be able to construct the p-adic inverse of any infinite prime. The correspondence of infinite primes with polynomials allows to construct infinite-P p-adics in a straightforward manner.

Consider first the infinite integers at the lowest level.

1. Infinite-P p-adics at the first level of hierarchy correspond to Laurent series like expansions using an irreducible polynomial P of degree n representing infinite prime. The coefficients of the series are numbers in the coefficient fields. Modulo p operation is replaced with modulo polynomial P operation giving a unique result and one can calculate the coefficients of the expansion in powers of P by the same algorithm as in the case of the ordinary p-adic numbers. In the case of n-variables the coefficients of Taylor series are naturally rational functions of at most $n-1$ variables. For infinite primes this means rationals formed from lower level infinite-primes.

2. Infinite-P p-adic units correspond to expansions of this type having non-vanishing zeroth order term. Polynomials take the role of finite integers. The inverse of a infinite integer in P-adic number field is obtained by developing the polynomial counterpart of $1/N$ in the following manner. Express N in the form $N = N_0(1+x_1P+..)$, where N_0 is polynomial with degree at most equal to $n-1$. The factor $1/(1+x_1P+...)$ can be developed in geometric series so that only the calculation of $1/N_0$ remains. Calculate first the inverse $\hat{N}_0^{-1}$ of N_0 as an element of the "finite field" defined by the polynomials modulo P: a polynomial having degree at most equal to $n-1$ results. Express $1/N_0$ as

$$\frac{1}{N_0} = \hat{N}_0^{-1}(1 + y_1P + ...)$$

and calculate the coefficients in the expansion iteratively using the condition $N \times (1/N) = 1$ by applying polynomial modulo arithmetics. Generalizing this, one can develop any rational function to power series with respect to polynomial prime P. The expansion with respect to a polynomial prime can in turn be translated to an expansion with respect to infinite prime and also mapped to a superposition of Fock states.

3. What about the norm of infinite-P p-adic integers? Ultra-metricity suggest a straightforward generalization of the usual p-adic norm. The direct generalization of the finite-p p-adic norm

would mean the identification of infinite-P p-adic norm as P^{-n}, where n corresponds to the lowest order term in the polymomial expansion. Thus the norm would be infinite for $n < 0$, equal to one for $n = 0$ and vanish for $n > 0$. Any polynomial integer N would have vanishing norm with respect to those infinite-P p-adics for which P divides N. Essentially discrete topology would result.

This seems too trivial to be interesting. One can however replace P^{-n} with a^{-n}, where a is any finite number a without losing the multiplicativity and ultra-metricity properties of the norm. The function space associated with the polynomial defined by P serves as a guideline also now. This space is naturally q-adic for some rational number q. At the lowest level the infinite prime defines naturally an ordinary rational number as the zero of the polynomial as is clear from the definition of the polynomial. At higher levels of the hierarchy the rational number is rational function of lower level infinite primes and by continuing the assignments of lower level rational functions to the infinite primes one ends up with an assignment of a unique rational number with a given infinite prime serving as an excellent candidate for a rational defining the q-adicity.

For the lowest level infinite primes the natural choice of a would be the rational number defined by it so that infinite-P p-adicity would indeed correspond to q-adicity meaning that number field property is lost.

7.2 Q-Adic Topology Determined By Infinite Prime As A Local Topology Of WCW?

Since infinite primes correspond to polynomials, infinite-P p-adic topology, which by previous considerations would be actually q-adic topology, is a natural candidate for a topology in function spaces, in particular in the WCW .

This view conforms also with the idea of algebraic holography. The sub-spaces of WCW can be modelled in terms of function spaces of rational functions, their algebraic extensions, and their P-adic completions. The mapping of the elements of these spaces to infinite rationals would make possible the correspondence between WCW and number theoretic anatomy of point of the imbedding space.

The q-adic norm for these function spaces is in turn consistent with the ultra-metricity for the space of maxima of Kähler functions conjectured to be all that is needed to construct S-matrix. Ultra-metricity conforms nicely with the expected four-dimensional spin glass degeneracy due to the enormous vacuum degeneracy meaning that maxima of Kähler function define the analog of spin glass free energy landscape. That only maxima of Kähler function would be needed would mean that radiative corrections to WCW integral would vanish as quantum criticality indeed requires. This TGD can be regarded as an analog of for an integrable quantum theory. Quantum criticality is absolutely essential for guaranteeing that S-matrix and U-matric elements are algebraic numbers which in turn guarantees number theoretic universality of quantum TGD.

7.3 The Interpretation Of The Discrete Topology Determined By Infinite Prime

Also $p = 1$-adic topology makes formally sense and corresponds to a discrete topology in which all rationals have unit norm. It results also results if one naively generalizes p-adic topology to infinite-p p-adic topology by defining the norm of infinite prime at the lowest level of hierarchy as $|P|_P = 1/P = 0$. In this topology the distance between two points is either 1 or 0 and this topology is the roughest possible topology one can imagine.

It must be however noticed that if one maps infinite-P p-adics to real by the formal generalization of the canonical identification then one obtains real topology naturally if coefficients of powers of P are taken to be reals. This would mean that infinite-P p-adic topology would be equivalent with real topology.

Consider now the possible interpretations.

1. At the level of function spaces infinite-p p-adic topology in the naive sense has a completely natural interpretation and states that the replacement of the Taylor series with its lowest term.

2. The formal possibility of $p = 1$-adic topology at space-time level suggests a possible interpretation for the mysterious infinite degeneracy of preferred extremals: one can add to any preferred extremal a vacuum extremal, which behaves completely randomly except for the constraints forcing the surface to be a vacuum extremal. This non-determinism is much more general than the non-determinism involving a discrete sequence of bifurcations (I have used the term association sequence about this kind of sequences). This suggests that one must replace the concept of 3-surface with a more general one, allowing also continuous association sequences consisting of a continuous family of space-like 3-surfaces with infinitesimally small time like separations. These continuous association sequences would be analogous to vacuum bubbles of the quantum field theories.

One can even consider the possibility that vacuum extremals are non-differentiable and even discontinuous obeying only effective $p = 1$-adic topology. Also Kähler-Dirac operator vanishes identically in this case. Since vacuum surfaces are in question, $p = 1$ regions cannot correspond to material sheets carrying energy and also the identification as cognitive space-time sheets is questionable. Since $p = 1$, the smallest possible prime in generalized sense, it must represent the lowest possible level of evolution, primordial chaos. Quantum classical correspondence suggests that $p = 1$ level is indeed present at the space-time level and might realized by the mysterious vacuum extremals.

8 How Infinite Primes Relate To Other Views About Mathematical Infinity?

Infinite primes is a purely TGD inspired notion. The notion of infinity is number theoretical and infinite primes have well defined divisibility properties. One can partially order them by the real norm. p-Adic norms of infinite primes are well defined and finite. The construction of infinite primes is a hierarchical procedure structurally equivalent to a repeated second quantization of a supersymmetric arithmetic quantum field theory. At the lowest level bosons and fermions are labelled by ordinary primes. At the next level one obtains free Fock states plus states having interpretation as bound many particle states. The many particle states of a given level become the single particle states of the next level and one can repeat the construction ad infinitum. The analogy with quantum theory is intriguing and I have proposed that the quantum states in TGD Universe correspond to octonionic generalizations of infinite primes.

It is interesting to compare infinite primes (and integers) to the Cantorian view about infinite ordinals and cardinals. The basic problems of Cantor's approach which relate to the axiom of choice, continuum hypothesis, and Russell's antinomy: all these problems relate to the definition of ordinals as sets. In TGD framework infinite primes, integers, and rationals are defined purely algebraically so that these problems are avoided. It is not surprising that these approaches are not equivalent. For instance, sum and product for Cantorian ordinals are not commutative unlike for infinite integers defined in terms of infinite primes.

Set theory defines the foundations of modern mathematics. Set theory relies strongly on classical physics, and the obvious question is whether one should reconsider the foundations of mathematics in light of quantum physics. Is set theory really the correct approach to axiomatization?

1. Quantum view about consciousness and cognition leads to a proposal that p-adic physics serves as a correlate for cognition. Together with the notion of infinite primes this suggests that number theory should play a key role in the axiomatics.

2. Algebraic geometry allows algebraization of the set theory and this kind of approach suggests itself strongly in physics inspired approach to the foundations of mathematics. This means powerful limitations on the notion of set.

3. Finite measurement resolution and finite resolution of cognition could have implications also for the foundations of mathematics and relate directly to the fact that all numerical approaches reduce to an approximation using rationals with a cutoff on the number of binary digits.

4. The TGD inspired vision about consciousness implies evolution by quantum jumps meaning that also evolution of mathematics so that no fixed system of axioms can ever catch all the mathematical truths for the simple reason that mathematicians themselves evolve with mathematics.

I will discuss possible impact of these observations on the foundations of physical mathematics assuming that one accepts the TGD inspired view about infinity, about the notion of number, and the restrictions on the notion of set suggested by classical TGD.

8.1 Cantorian View About Infinity

The question which I have but repeatedly under the rug during the last fifteen years concerns the relationship of infinite primes to the notion of infinity as Cantor and his followers have understood it. I must be honest: I have been too lazy to even explain to myself what Cantor really said. Therefore the reading of the New Scientist article "The Ultimate logic: to infinity and beyond" (see `http://tinyurl.com/3va48jq`) [A16] was a pleasant surprise since it gave a bird's eye of view about how the ideas about infinity have evolved after Cantor as a response to severe difficulties in the set theoretic formulation for the foundations of Mathematics.

8.1.1 Cantor's paradize

I try to summarize Cantor's view about infinity first. Cantor was the pioneer of set theory, in particular the theory of infinite sets. Cantor started his work around 1870. His goal was to formulate all notions of mathematics in terms of sets, in particular natural numbers. Cardinals and ordinals define two kind of infinite numbers in Cantor's approach.

1. Cantor realized that real numbers are "more numerous" than natural numbers and understood the importance of one-to-one correspondence (bijection) in set theory. One can say that two sets related by bijection have same cardinality. This led to the notion of cardinal number. Cardinals are represented as sets and two cardinals are same if a bijection exists between the corresponding sets. For instance, the infinite cardinals assignable to natural numbers and reals are different since no bijection between them exists.

2. The definition of ordinal relies on successor axiom of natural numbers generalized to allow infinitely large ordinals. Given ordinal can be identified as the union of all ordinals strictly smaller than it. Well ordering is a closely related notion and states that every subset of ordinals has smallest element. One can classify ordinals to three types: 0, elements with predecessor, and elements without predecessor such as ω, which corresponds to the ordinal defined as the union of all natural numbers.

   The number of ordinals much larger than the number of cardinals. This is clear since the notion of ordinal involves additional structure coming from their ordering. A given cardinal corresponds to infinitely many ordinals and one can identify the cardinal as the smallest ordinal of this kind. For instance, ω and $\omega + n$ correspond to same cardinal α_0 (countable infinity) for all finite values of n.

3. Cantor introduced the notion of power set as the set of all subsets of the set and proved that the cardinality of the power set is larger than that of set. Cantor introduced also the continuum hypothesis stating that there are no cardinals between the cardinal $\aleph_0$ *resp.* $\aleph_1$ assignable to natural numbers *resp.* reals. Hilbert represented continuum hypothesis as one of his 23 problems in his talk at the 1900 International Congress of Mathematicians in Paris. Hilbert was also a defender of Cantor and introduced the term Cantor's paradise.

4. Cantor developed the arithmetics of ordinals based on sum, product, and power: each of these operations is expressible in terms of set theoretic concepts. For infinite ordinals multiplication and sum are not commutative anymore. This looks highly counter intuitive and requires detailed definition of the sum and product. Sum means just writing the ordered sequences representing ordinals in succession. To see the non-commutativity of sum it is enough to notice that the number of elements having predecessor is not the same for $\omega + n$ and $n + \omega$.

To see the non-commutativity of product it is enough to notice that the product is define as cartesian product $S \times T$ of the ordered sets representing the ordinals. This means that every element of T is replaced with S. It is easy to see that $n \times \omega$ and $\omega \times n$ are different.

One can define also the powers (exponentials) in the arithmetics of ordinals: exponent must reduce to the notion of power set X^Y, which can be realized as the set of maps $Y \rightarrow X$ and has formally $\#X^{\#Y}$ elements.

It is pity that the we physicists have so pragmatic attitude to mathematics that we do not have time to realize the beauty of the idea about reduction of all mathematics to set theory. This is even more regrettable since it might well be that the manner to make progress in physics might require replacing the mathematics with a mathematics which does not rely on set theory alone.

8.1.2 Snakes in Cantor's paradise

Cantor's paradise is extremely beautiful place but there are snakes there. Continuum hypothesis looked to Cantor intuitively obvious but the attempts to prove it failed. Betrand Russel showed in 1901 that the logical basis of Cantor's set theory was flawed. This manifested itself via a simple paradox. Assume that it makes sense to speak about the set of all ordinals. This is by definition ordinal itself since ordinal is a set consisting of all ordinals strictly smaller than it. But this would mean that the set of all ordinals is its own member! The famous barber's paradox is a more concrete manner to express Russel's antinomy. One cannot speak of the set of ordinals and must introduce the notion of class. Russell introduced also the notion of types and type theory.

At 1920 Ernst Zermelo and Abraham Fraenkel devised a series of rules for manipulating sets but these rules did not allow to resolve the status of the continuum hypothesis. The stumbling block was the rule known as "axiom of choice" stating that if you have a collection of sets you can form a new set by picking one element from each of them. At first this sounds rather obvious but in the case when there is no obvious rule telling how to do it, situation becomes non-trivial. Then Polish mathematicians Stefan Banach and Alfred Tarski managed to show how the axiom would allow the division of a spherical ball to six subsets which can then be arranged to two balls with the same size as the original ball using only rotations and translations. These six sets are non-measurable in terms of Lebesque measure. The non-intuitive outcome must relate to the definition of the volume of the ball that is integration or measure theory: the axioms of measure theory should bring in constraints preventing construction of the six sets.

Around 1931 Kurt Gödel proved the incompleteness theorem that it is not possible to axiomaticize arithmetics using any axiom system. There always remain unprovable propositions, which are true and cannot be proved to be true. This kind of statement is analogous to "I am a statement which cannot be proved to be true". If this statement could be proved to be true it would not be true.

8.1.3 Constructing logical universes

The attempts to expel the snakes from Cantor's paradise led to the idea that by posing some constraints it might be possible to construct logically consistent set theory obeying Zermelo-Fraenkel axioms such that continuum hypothesis and the axiom of choice would hold true and which would be free of paradoxes such as Banach-Tarski paradox.

Around 1938 Gödel introduced what he called constructible Universe (see `http://tinyurl.com/y43jun`) or L world satisfying these constraints. The structure of L world is hierarchical and one can say that the successor idea manifests itself directly in the construction. The levels are labeled by ordinals and one can always add a new level. The introduction of a new level to the hierarchy means that new axioms are introduced to the system bringing in meta level to the mathematical structure. The axiom system can be extended indefinitely. Gödel's theorem holds true at given level of hierarchy but by adding new levels non-probable truths can be made provable.

1963 Paul Cohen however demonstrated that there is infinite number of this kind of L worlds. In some of them continuum hypothesis holds true, in some of them the number of cardinals between $\aleph_0$ andα_1 can be arbitrary large - even infinite. This initiated a boom of constructions brings in mind the inflation of GUTs in particle physics and the endless variety of brane constructions and the landscape misery of M-theory. From the point of view of physicist the non-uniqueness in foundations of mathematics does not seem to matter much since the everyday mathematics would remain the familiar one. One can of course ask what about quantum theory: should quantum physics replace classical physics in the formulation of fundamental fo mathematics.

For instance, von Neumann (see `http://tinyurl.com/lkndbqo`) proposed one particular L world. In von Neumann unverse one starts from natural numbers and constructs its power set and at each step in the construction one consideres power set assigned to the sete obtained at the previous level. It is clear that one imagine several options. One could consider all subsets, only finite subsets, or only subsets which have cardinality smaller than the set itself. Power sets identified as the set of all finite subsets would give minimal option. Power set identified as the set of all subsets would give the maximal option.

The work of Hugh Woodin represented in 2010 International Congress of Mathematicians in Hyderabad, India represents the last twist in the story. Woodin argues that one must step outside the system that is conventional mathematical world to solve the problem. Woodin has introduced so called Woodin cardinals whose existence implies that all "projective" subsets of reals have a measurable size: it is not an accident that the word "measure" appears here when one recalls what Banach-Tarski paradox states. Woodin was motivated by the problems of set theory. He expresses this by saying "Set theory is riddled with unsolvability. Almost any problem of set theory is unsolvable".

Woodin proposed his own constructive universe which he calls ultimate L. It has all the desired properties: in particular, continuum hypothesis holds true. Physicists reader need not get frustrated if he fails to intuit why this is the case: for a decade ago Wooding himself did not believe in this. Also this L world is infinite tower to which one can add new levels.

8.2 The Notion Of Infinity In TGD Universe

The construction of infinite primes, integers, and rationals brings strongly in mind the L worlds of Gödel and followers and this inspires the idea about concrete comparison of these approaches to see the differences.

8.2.1 Rule of thumb

It is good to start with a rule of thumb allowing to make strong conclusions about the cardinalities of infinite primes. If one considers the set formed by all finite subsets of a countable set you get a countable set because these subsets can be expressed as bit sequences with finite number if non-vanishing binary digits telling whether given element of set belongs to the subset or not: this bit sequence corresponds to a unique integer. If *all* subsets (also infinite) are allowed the set is not countably finite. If continuum hypothesis holds true it has at least as many elements as real line.

2-adic integers are good example. Consider first all 2-adic numbers with a *finite* number of non-vanishing bits (finite as real numbers). You get a countably infinite set since you can map these bit sequences to natural numbers in an obvious manner.

Consider next all possible bit sequences: most of them have infinite number bits. These numbers form naturally 2-adic continuum with 2-adic topology and differentiability. 2-adics can be mapped to real continuum in simple manner: canonical identification allows to do this continuously. The cardinality of these bit sequences is same as for reals as the rule of thumb would predict.

The hierarchy of infinite integers is based on number theoretical view about infinity and it would seem that these infinities are between the countable infinity and infinity defining the number of points of real axis. This reflects the fact that number theoretic infinity is much more refined notion than the infinities associated with cardinals and even ordinals. For instance, one can divide these infinities whereas Cantorian arithmetics contains only sum, product and power.

8.2.2 How Cantor's ordinals relate to the construction of infinite primes?

The fascinating question is whether the comparison of the construction of infinite primes, integers and rationals could relate to the work of Cantor and Gödel and his followers could provide new insights about infinite primes themselves.

1. What is intriguing that L-worlds are defined as infinite hierarchies just as the hierarchy of infinite primes and associated hierarchies. The great idea is that these constructions are

essentially set theoretic in accordance with the vision that mathematics should reduce to set theory. As already noticed, naive set theory however leads to paradoxes which motivates the work of Gödel and followers. The basic physical philosophy is the identification of physical state as a set: this is essentially a notion belonging to classical physics.

2. TGD approach is algebraic rather than set theoretic. The construction is based on explicit formulas assuming the existence of weird quantities defined as product of all primes at previous level. These quantities can be taken as purely algebraic notions without any attempt to find a set theoretic definition.

   The possibility to interpret the construction as a repeated second quantization of a supersymmetric arithmetic quantum field theory with bosons and fermions labeled by ordinary primes at the loweset level of hierarchy replaces the set theoretic picture. These weird products of all primes represent Dirac sea at a given level of hierarchy and the many particles states of previous level become elementary particles at the new level of hierarchy. This construction is proposed to have a direct physical realization in terms of many-sheeted space-time and generalized to the level of octonionic primes is suggested to allow number theoretic interpretation of standard model quantum numbers.

   Perhaps it is not mere arrogance of quantum physics to argue that the classical set theoretic view about physical state is replaced with quantum view about it. Algebra replaces set theory and real and p-adic topologies are essential: for instance, infinite primes are infinite only in real topology.

One can raise many interesting questions. Although the underlying philosophies are very different, one can ask whether it might be possible to reduce TGD inspired construction to set theory playing key role in the construction of ordinals?

1. Can one assign to a given infinite integer a set in a natural manner? At the lowest level of hierarchy infinite prime can be mapped to a rational. Could one assign to this rational a set in cartesian product $N \times N$? Does this argument generalize to higher levels? Could the construction discussed in [K9] allow to realize the set theoretic representation?

2. The notion of divisibility and explicit formulas for infinite integers obviously imply that the number of infinite numbers is much larger than cardinals of Cantor. This is true also for the ordinals of Cantor. How infinite integers relate to the ordinals of Cantor for which successor axiom is true? Also now it makes sense to form successors and in general they correspond to products of infinite primes which can be mapped to polynomials of several variables. For infinite integers however also the predecessor always exists. For instance $X \pm 1$ are infinite primes, where X represents the product of primes at previous level. Only zero fails to have predecessor for infinite natural numbers.

3. In TGD framework one loses the very essential notion of well-orderedness stating that every ordinal corresponds to a set with smallest element: that is element without predecessor. For instance, the infinite numbers known as limits and by definition are infinite and have no predecessor, the simplest example about limit is ω, which corresponds to the union of all natural numbers. The study of predecessors allowed to conclude that the sum and product are non-commutative for ordinals. Since the notion of well-ordered set does not make sense for infinite integers, one cannot identify infinite integers as ordinals.

   One must however remember that just the well-orderedness hypothesis together with successor axiom allows to express ordinal as a union of strictly smaller ordinals. This in turn leads to the conclusion that ordinals cannot form a set and to Russel's antinomy and are responsible for the many problems of set theory forcing Wooding to sigh "Set theory is riddled with unsolvability. Almost any problem of set theory is unsolvable". Maybe the well-orderedness axiom is simply too strong for infinite ordinals.

4. Sum, product, and power are the basic operations in the arithmetics of ordinals. All they reduce to set theoretic constructions. One can however define these operations purely algebraically. The algebraic definition of sum and product makes sense since one can map the infinite integers to polynomials of several variables. The possibly existing set theoretic

definition of infinite integers using infinite sets cannot be consistent with the commutativity of sum and product defined algebraically. Either algebra or set theory but not both!

5. Also the notion of power makes sense for ordinals and relies on the notion of power set. Could the algebraic definition of exponential make sense? If the exponent N of M^N is finite integer, then the exponent makes sense for infinite M. If N is infinite integer it does not. Hence it seems that the analogs of numbers like ω^ω do not exist in TGD inspired L universe.

6. The failure of set theoretic reductionism brings in mind the motivic approach to integration as purely algebraic approach applied to the symbol defining the integral instead of a number approach based on set theoretic notions. The motivation of the motivic approach in p-adic context is that p-adic numbers are not well-ordered so that one loses the notion of boundary and orientation as topological concepts although they can make sense algebraically.

For the hierarchy infinite integers the notion of infinity relies on real norm, which is essentially length rather than on the cardinality of a set. This infinity is essentially non-Cantorian and it is perhaps useless to try to relate it to that of ordinal or cardinal. There is just an infinite hierarchy of infinities which replaces the hierarchy of ordinals and for which the real norm of ratio makes possible partial ordering. Clearly the notion of infinity is extremely slippery and one must carefully specify what one means with infinite.

8.2.3 Cardinals in TGD Universe

What about cardinals in TGD framework? There seems to be no reason for giving them up and the first guess is that TGD replaces cardinals and ordinals of Cantor with cardinals and the hierarchy of infinite primes, integers, and rationals.

1. The first question is what is the cardinal assignable to infinite primes at the first level of hierarchy. For the set of finite primes the cardinal is $\aleph_0$. For the first level of infinite primes the situation is not so simple. The simple infinite primes correspond to free Fock states constructed from fermions and bosons labelled by primes. They are in one-one correspondence with rationals. There is however infinite number of many particle bound states representable as products of irreducible polynomials of one variable with integer coefficients and having finite number of roots which are algebraic numbers. The set of algebraic numbers is countable. This suggests that the cardinality of set of infinite primes at the first level of hierarchy corresponds to $\aleph_0$. This if course assuming that infinite integers and rationals for a set although they themselves cannot be described as sets.

   If one allows Fock states containing infinite number of particles and having thus infinite energy one obtains formally polynomials of infinite degree identifiable as Taylor expansions. In this case the roots can be transcendental numbers and one expects that a cardinal larger than $\aleph_0$, say $\aleph_1$ emerges. In von Neumann's Universe one indeed allows all subsets and $\aleph_1$ appears already at the first level. The higher cardinals appear at higher levels.

   One cannot exclude the Fock states containing infinite number of quanta if one accepts the idea that infinite prime representing quantum state characterizing entire Universe make sense. Does this mean that $\aleph_1$ has meaning only for entire universe and for states carrying infinite energy (in ZEO the positive energy part of zero energy state would carry the infinite energy)?

2. What happens at the next levels of the hierarchy? One possibility is that infinite primes at each level define a countable set. The point is that in polynomials representation one considers only finite degree polynomials depending on finite number of variables, having rational coefficients. Therefore everything at the level of definitions is countable and finite and the product X of primes of previous level is just an algebraic symbol identifiable as a variable of polynomial.

3. In an alternative construction of infinite integers suggested in [**?**]ne considers the first level of the hierarchy the set of finite subsets of algebraic numbers and the set of finite subsets of this set at the next level and so on. All these sets are countable which suggests that the number of infinite primes at each level of the hierarchy is countable and that only the completion of

algebraic number to reals or p-adic can give rise to $\aleph_1$. This would conform with the fact that quantum physics is basically based on counting of quanta and that finite measurement resolution is an essential restriction on what we can know.

8.2.4 What about the axiom of choice?

Axiom of choice has several variants. One variant is axiom of countable choice. Second variant is generalized continuum hypothesis states that the cardinality of an infinite set is between that of infinite set S and its power set: in other words there is no cardinal satisfying $\aleph_\alpha < \lambda < 2^{\aleph_\alpha}$ or equivalently: $\aleph_{\alpha+1} = 2^{\aleph_\alpha}$. For a finite collection of sets it can be proved but already when on has a countable collection of nonempty set and in the case that one cannot uniquely specify some preferred element of each set, axiom of choice must be postulated. For instance, each subset of natural numbers has smallest element so that there is no need to postulate axiom of choice separately. Also closed intervals of real axis have smallest element.

What happens to the axiom of choice in TGD Universe. TGD is a physical theory and this means that the laws of classical physics strong considerations on the allowed sets. Classical physics is in TGD framework the dictated by the Kähler action and by a principle selecting its preferred extremals. Although several almost formulations for this principle exist, it is far from being well-understood and it is not clear whether one can give explicit formula for preferred extremals. One formulation is as quaternionic sub-manifolds of 8-D imbedding space allowing octonionic structure in its tangent space and defined by octonionic representation of the gamma matrices defining the Clifford algebra.

1. The world of classical worlds can be regarded as the space of preferred extremals of Kähler action identifiable as certain 4-surfaces in $M^4 \times CP_2$. The mere extremal property implies also smoothness of the partonic 2-surfaces so that very powerful constraints are involved: therefore situation is very far from the extreme generality of set theory where one does assumes neither continuity nor smoothness. Zero energy ontology means that this space effectively reduces to a collection of spaces assignable to causal diamonds. Strong form of holography reduces this space effectively to the space consisting of collectinons of partonic 2-surfaces at the light-like boundaries of CD plus 4-D tangent space data at them which very probably cannot be chosen freely.

2. In this kind of situation it might well happen that all collections of sets, say are finite or in the case that that they are countable they allow a unique choice of preferred point. Axiom of choice would not be needed. The specification of a preferred point of every 4-surface in the collection does not look a problem for a pragmatic physicist, since one can restrict the consideration to the boundaries of causal diamonds and consider for instance minimum of light-like radial coordinate. In fact, finite measurement resolution leads to the effective replacement of partonic 2-surfaces with the collection of ends of braid strands and the ends of braid strands define the preferred points. One might say, that physics with finite measurement resolution performs the choice automatically. A stronger form of this choice is that the points in question are rational points for a natural choice of the imbedding space coordinates.

8.2.5 Generalization of real numbers inspired by infinite integers

Surreal numbers define a generalization of reals obtained by introducing a hierarchy of infinite reals and infinitesimals as their inverses. Infinite integers and rationals in TGD sense could give rise to a similar generalization so that one would have an infinite hierarchy of 8-D imbedding space such that at given level previous level would represent infinitesimals.

TGD suggests another generalization of reals. One can construct from infinite integers rationals with unit norm. A possible interpretation would be as zero energy states with denominator and numerator representation positive and negative energy parts of the zero energy state and vanishing of total quantum numbers represented by real unit property. These numbers would have arbitrarily complex number theoretical anatomy however.

This structure has enormous representative power and one could dream that the world of classical worlds and spinor fields in this space could allow representation in terms of these real units.

Brahman Atman Identity would be realized: the structure of single space-time point invisible to ordinary physics would represent the world of classical worlds! Single space-time point would be the Platonia!

Could one say that the space of all infinite rationals which are real units is countable? If previous arguments are correct this would seem to be true. If this is true, then TGD inspired notion of infinity would be extremely conservative as compared to the view proposed by Cantor and his followers using the Cantorian criteria. Just $\aleph_n$, $n = 0, 1$ and hierarchy of infinite integers which are countable sets. One can of course, ask how many surfaces WCW contains, what $\aleph$ is in question. This depends on the properties of preferred extremals. If partonic 2-surfaces can be chosen freely at the boundaries of CDs the restrictions come only from smoothness of the imbedding of the partonic 2-surfaces and tangent space data. The space of all functions from reals to reals has cardinality $2^{\aleph_1}$ which suggests that the cardinality is not larger than this, perhaps smaller since continuity and smoothness poses strong conditions. The natural guess is that the tangent space of WCW can be modelled as and infinite-dimensional separable Hilbert space which has cardinality $\aleph_1$.

TGD leads also a second generalization of the number concept motivated by number theoretical universality inspiring the attempt to glue different number fields (reals and various p-adics) together among common numbers -rationals in particular- to form a larger structure [K10].

To sum up, the distinctions between Cantorian and TGD inspired approaches are clear. Cantorian approach relies on set theory and TGD on number theory. What is common is the hierarchy of infinities.

8.3 What Could Be The Foundations Of Physical Mathematics?

Theoretical physicists do not spend normally their time for questioning the foundations of mathematics. They calculate. There are exceptions: Von Neumann was both a theoretical physicist developing mathematical foundations of quantum theory and mathematician building the mathematics of quantum theory and also proposing his own L world for foundations of mathematics.

A physicist posing the question "What should be done for the foundations of mathematics?" sounds blasphemous and the physicist should add the attribute "physical" to "mathematics" to avoid irritation. In any case, the fact is that the problems plaguing set theory and therefore the foundations of mathematics had been discovered roughly century ago and no commonly accepted solution to these problems have been found. The foundations of mathematics rely on classical physics and quantum view about existence suggests that the foundations of mathematics might need a revision.

Again the work of von Neumann comes readily into mind. The goal of von Neuman was to build a non-commutative measure theory: the outcome was the three algebras bearing his name and defining the mathematical backbone of three kinds of quantum theories. Factors of type I are natural for wave mechanism with finite number of degrees of freedom. In QFT hyperfinite factors of type III appear. In TGD framework hyperfinite factors of type II (and possibly of type III) are natural.

Connes who has studied von Neumann algebras highly relevant to quantum physics proposed the notion of non-commutative geometry in terms of a spectral triplet defined by C^* algebra A, Hilbert space H, and Dirac operator D with some additional properties. As a special case one re-discovers Riemannian manifolds using commutative function algebra, the Hilbert space of continuous functions, and certain kind of Dirac operator.

Physicists are usually mathematical opportunists and do not want to use time to ponder the foundations of mathematics My belief is that physicists should get rid of this attitude and make fool of themselves by posing the childish questions of physicist in the hope that some real mathematician might get interested. In order to not irritate mathematicians too much I will talk about physical mathematics instead of mathematics in the sequel.

The proposal that infinite primes, integers, and rationals should replace Cantor's ordinals and surreal numbers [K9] has been already made. This would allow to get rid of Russell's antinomy, leave the notion of cardinal intact. Also axiom of choice looks too strong from the point of view of physics.

8.3.1 Does it make sense to speak about physical set theory?

For the physicist set theory looks quite too general. In the recent day physical theories sets are typically manifolds, sub-manifolds, or orbifolds. Feynman diagrams represent example of 1-D singular manifolds and in TGD generalized Feynman diagrams of TGD fail to be 3-manifolds only at the vertices represented as 2-D partonic surfaces. In string theories and in twistor approach to gauge theories algebraic geometry is important. Branes are typically algebraic surfaces. The spaces are endowed with various structures: besides metric induced topology one differential structure, differential forms, metric, spinor structure, complex and Kähler structure, etc...

1. In algebraic geometry sets are replaced with varieties and basic set theoretic operations such as intersection and union are algebraized. Physicists should not fail to realize how profound this algebraization of the set theory is. The price that must be paid is that varieties are manifolds only locally. What limitations does this mean for set theory? Is it enough to formulate set theory algebraically? In TGD framework this could be possible in the intersection of real and p-adic worlds (WCW s) since set theoretic operations would have algebraic representation. For instance, $A \subset B$ would be formulated by adding additional functions for which the intersection of zero locus with B defines A.

   The algebraic notion of set as a variety is extremely restrictive: maybe the problems of set theory are partially due to the neglect of the fact that allowed sets must have a physical realization. Every physicists of course has her own pet theory, which he regards as the real physics, and one natural condition on any acceptable physics is that it can emulate sufficiently general spaces - to act as a kind of mathematical Turing machine. At least real and complex manifolds with arbitrary dimension should have some kind of physical representation. One can imagine this kind of representation in terms of unions of partonic 2-surfaces since union can be regarded also as a Cartesian product as long as the surfaces do not intersect.

2. The introduction of topology is the first step in bringing structure to the set theoretic primordial chaos. Metric topology is standard in physics at space-time level. More refined topologies can be certainly found in highly technical mathematical physics articles. In algebraic geometry Zariski topology is important but has its problems realized by Groethendienck in his attempts to build a universal cohomology theory working in all number fields. The closed sets of Zariski topology are varieties. Their complements would be open sets open also in norm based topology. Zariski topology is obviously much rougher than the metric topology. Zariski topology makes sense also for p-adic number fields. This kind of topology might make sense in TGD framework if one restricts the consideration to the intersection of real and p-adic worlds identified at the level of WCW as the space of algebraic surfaces defined using polynomials with rational coefficients and having finite degree.

3. In TGD framework preferred extremals of Kähler action define space-time surfaces and strong form of holography makes the situation effectively 2-dimensional. The conjecture is that preferred extremals correspond to quaternionic surfaces of octonionic 8-space. Octonionic structure is associated with the octonionic representation of the imbedding space gamma matrices (not actually matrices any more!) defining the Clifford algebra. Associativity would be the basic dynamical principle. Does this mean that number theory- in particular classical number fields- should appear in the formulation of the foundations of physical mathematics? This idea is attractive even when one does not assume that TGD Universe is the Universe.

What is beautiful that algebraic geometry brings in also number theory. One might hope that the foundations of physical mathematics could be based on the fusion of set theory, geometry, algebra, and number theory.

8.3.2 Quantum Boolean algebra instead of Boolean algebra?

Mathematical logic relies on the notion of Boolean algebra, which has a well-known representation as the algebra of sets which in turn has in algebraic geometry a representation in terms of algebraic varieties. This is not however attractive at space-time level since the dimension of the algebraic variety is different for the intersection *resp.* union representing AND *resp.* OR so that only only finite number of ANDs can appear in the Boolean function. TGD inspired interpretation of the fermionic sector of the theory in terms of Boolean algebra inspires more concrete ideas about the realization of Boolean algebra at both quantum level and classical space-time level and also suggests a geometric realization of the basic logical functions respecting the dimension of the representative objects.

1. In TGD framework WCW spinors correspond to fermionic Fock states and an attractive interpretation for the basis of fermionic Fock states is as Boolean algebra. In zero energy ontology one consider pairs of positive and negative energy states and zero energy states could be seen as physical correlates for statements $A \rightarrow B$ or $A \leftrightarrow B$ with individual state pairs in the quantum superposition representing various instances of the rule $A \rightarrow B$ or $A \leftrightarrow B$. The breaking of time reversal invariance means that either the positive or negative energy part of the state (but not both) can correspond to a state with precisely define number of particles with precisely defining quantum numbers such as four-momentum. At the second end one has scattered state which is a superposition of this kind of many-particle states. This would suggest that $A \rightarrow B$ is the correct interpretation.

2. In quantum group theory (see `http://tinyurl.com/3tors5`) [A6] the notion of co-algebra (see `http://tinyurl.com/27dkk4y`) [A2] is very natural and the binary algebraic operations of co-algebra are in a well-defined sense time reversals of those of algebra. Hence there is a great temptation to generalize Boolean algebra to include its co-algebra (see `http://tinyurl.com/y8s585p8`) [A17] so that one might speak about quantum Boolean algebra. The vertices of generalized Feynman diagrams represent two topological binary operations for partonic two surfaces and there is a strong temptation to interpret them as representations for the operations of Boolean algebra and its co-algebra.

   (a) The first vertex corresponds to the analog of a stringy trouser diagram in which partonic 2-surface decays to two and the reversal of this representing fusion of partonic 2-surfaces. In TGD framework this diagram does not represent classically particle decay or fusion but the propagation of particle along two paths after the decay or the reversal of this process. The Boolean analog would be logical OR ($A \vee B$) or set theoretical union $A \cup B$ *resp.* its co-operation. The partonic two surfaces would represent the arguments (*resp.* co-arguments) A and B.

   (b) Second one corresponds to the analog of 3-vertex for Feynman diagram: the three 3-D "lines" of generalized Feynman diagram meet at the partonic 2-surface. This vertex (co-vertex) is the analog of Boolean AND ($A \wedge B$) or intersection $A \cap B$ of two sets *resp.* its co-operation.

   (c) I have already earlier ended up with the proposal that only three-vertices appear as fundamental vertices in quantum TGD [K1]. The interpretation of generalized Feynman diagrams as a representation of quantum Boolean algebra would give a deeper meaning for this proposal.

   These vertices could therefore have interpretation as a space-time representation for operations of Boolean algebra and its co-algebra so that the space-time surfaces could serve as classical correlates for the generalized Boolean functions defined by generalized Feynman diagrams and expressible in terms of basic operations of the quantum Boolean algebra. For this representation the dimension of the variety representing the value of Boolean function at classical level is the same as as the dimension of arguments: that is two. Hence this representation is not equivalent with the representation provided by algebraic geometry for which the dimension of the geometric variety representing $A \wedge B$ and $A \vee B$ in general differs from that for A and B. If one however restricts the algebra to that assignable to braid strands, statements would correspond to points at partonic level, so that one would have discrete sets and the set theoretic representation of quantum Boolean algebra could make sense. Discrete sets are indeed the only possibility since otherwise the dimension of intersection and union are different if algebraic varieties are in question.

3. The breaking of time reversal invariance is accompanied by a generation of entropy and loss of information. The interpretation at the level of quantum Boolean algebra would be following.

The Boolean functionandOR assign to two statements a single statement: this means a gain of information and at the level of physics this is indeed the case since entropy is reduced in the process reducing the number of particles. The occurrence of co-operations ofandOR corresponds to particle decays and uncertainty about the path along which particle travels (dispersion of wave packet) and therefore loss of information.

(a) The "most logical" interpretation for the situation is in conflict with the identification of the arrow of logic implication with the arrow of time: the direction of Boolean implication arrow and the arrow of geometric time would be opposite so that final state could be said to imply the initial state. The arrow of time would weaken logical equivalence to implication arrow.

(b) If one naively identifies the arrows of logical implication and geometric time so that initial state can be said to imply the final state, second law implies that logic becomes fuzzy. Second law would weak logical equivalence to statistical implication arrow.

(c) The natural question is whether just the presence of both algebra and co-algebra operations causing a loss of information in generalized Feynman diagrams could lead to what might be called fuzzy Boolean functions expressing the presence of entropic element appears at the level of Boolean cognition.

4. This picture requires a duality between Boolean algebra and its co-algebra and this duality would naturally correspond to time reversal. Skeptic can argue that there is no guarantee about the existence of the extended algebra analogous to Drinfeld double (see `http://tinyurl.com/y7tpshkp`) [A15] that would unify Boolean algebra and its dual. Only the physical intuition suggests its existence.

These observations suggest that generalized Feynman diagrams and their space-time counterparts could have a precise interpretation in quantum Boolean algebra and that one should perhaps consider the extension of the mathematical logic to quantum logic. Alternatively, one could argue that quantum Boolean algebra is more like a model for what mathematical cognition could be in the real world.

8.3.3 The restrictions of mathematical cognition as a guideline?

With the birth of quantum theory physicists ceased to be outsiders since it was impossible to consider quantum measurement as something not affecting the measured system in any way. With the advent of consciousness theory physicists have been forced to give up the idea about unidirectional action with with reality and have become a part of quantum Universe - self. This also requires dramatic modification of the basic ontology forcing to give up the physicalistic dogmas. Consciousness involves free will manifested in ability to select and create something completely new in each quantum jump. Physical Universe is not given but is re-created again and again and evolves.

In standard mathematics mathematician is still a complete outsider, and the possible limitations of mathematical cognition are not considered seriously in the attempts to formulate the foundations of mathematics. Mathematicians still choose effortlessly one element from each set of infinite collection of sets. We know that in numerics one is always bound to introduce cutoff on the number of bits and use finite subset of rational numbers but also this has not been taken into account in the formulation of foundations as far as I know. If one takes consciousness theory seriously one is led to wonder what are the physical restrictions on mathematical cognition and therefore on physical mathematics. What looks obvious that the idea about mathematics based on fixed axiomatics must be given up. The evolution of the physical universe and of consciousness means also the evolution of (at least physical) mathematics. The paradox of self reference plaguing conventional view about consciousness and leading to infinite regress disappears when this regress is replaced with evolution.

Suppose that life resides and cognitive representations are realized in the intersection of real and p-adic worlds reducing to intersections of real and p-adic variants of partonic 2-surfaces at space-time level. At the level of WCW the intersection of real and p-adic worlds could correspond to the space of partonic 2-surfaces defined by rational functions constructed using polynomials of finite degree with rational coefficients.

What kind of restrictions of this picture poses set theory, topology, and logic? The reader can of course imagine restrictions on some other fields of mathematics involved. The question in the case of the set theory and topology has been already touched. In the case of logic the key question seems to concern the operational meaning of $\forall$ and $\exists$, when the finite resolution of measurements and cognitive representation are taken into account. What these universal quantors really mean: what is their domain of definition?

Consider first the domain of definition at space-time level.

1. Should all theorems be formulated using $\forall$ and $\exists$ restricted to the dense subset rationals of 8-D imbedding space. Since continuous function is fixed from its values in a dense subset, this assumption is not so strong unless there are other restrictions.

2. At space-time surface and partonic 2-surfaces the situation is different. The assumption that only the common rational points of real and p-adic surfaces define cognitive representations poses a strong limitation since typically the number of rational points of 2-surface is expected to be finite. Algebraic extensions of p-adic numbers extend the number of common points and one can imagine an evolutionary hierarchy of mathematics realized in terms of geometry of partonic 2-surfaces reflecting itself as the geometry of space-time surfaces by strong form of holography.

3. The orbits of the rational points selected at the ends of partonic 2-surfaces are braids along light-like 3-surfaces. At space-time level one has world sheets or strings which form in general case 2-braids. This picture leads to a what I have used to call almost topological QFT.

What about the domain of definition of existence quantors at the level of WCW ? The natural conjecture is that the surfaces in the intersection of real and p-adic worlds form a dense set of full WCW so that everything holding true in the intersection would hold true generally and one coul dhope that systems which are living in the proposed sense are able to discover interesting mathematics.

Suppose that the partonic 2-surfaces decompose into patches such that in each patch the surface is a zero locus of polynomials with rational coefficients. Since polynomials can be seen as Taylor series with cutoff one can hope that they form a dense subset. Since rationals are dense subset of reals, one can hope that also the restriction to rational coefficients preserves the dense subset property and living subsystems are able to represent all that is needed and completion takes care of the rest as it does for rationals. The notion of completion leading from rationals to various algebraic numbers fields and also to reals and complex numbers would become the fundamental principle leading from number theory to metric topology.

Physicist reader has certainly noticed that "rational point" does not represent a general coordinate invariant notion.

1. The coordinates of point are rational in preferred coordinates and the symmetries of the 8-D imbedding space suggest families of preferred coordinates. The moduli space for CDs would be characterized by the choice of these preferred coordinates dictating also the choice of quantization axes so that quantum measurement theory would be realized as a decomposition of WCW to a union corresponding to different choices. State function reduction would involve also a localization determining quantization axes.

2. There are many possible choices of quantization axes/preferred coordinates and this means a restriction of general coordinate invariance from group of all coordinate transformations to a discrete subgroup of isometries which is not unique. Cognition would break the general coordinate invariance. The world in which the mathematician thinks using spherical coordinates differs in some subtle manner from the world in which she thinks using Cartesian coordinates. Mathematician does not remain outside Platonia anymore just as quantum physicists is not outside the physical Universe!

Axiom of choice relates to selection, which can be regarded as a cognitive act. The question whether axiom of choice is needed at all has been already discussed but a couple of clarifying comments are in order.

1. At quantum level selection would be naturally assigned with state function reduction, also the state function reduction selecting quantization axes. The cascade of state function reductions - starting from the scale of CD and proceeding fractally downwards sub-CD by sub-CD and stopping when only negentropic entanglement stabilized by NMP remains - could be how Nature performs the choice. State function reduction would involve also the choice of quantization axes dictating possible subsequent choices. Note that non-deterministic element would be involved with the quantum choice.

2. If life and cognitive representations are at the intersection of real and p-adic worlds, it would seem that rational points are chosen at space-time level and algebraic 2-surfaces at WCW level. As explained, it is easy to imagine the collection of sets from which one selects points is always finite or that there is a natural explicit criterion allowing to select preferred point from each set. Finite measurement resolution implying braids and string world sheets could provide this criterion. If so, the axiom of choice would be un-necessary in physical mathematics. Finite measurement resolution suggests that for partonic 2-surfaces the ends of braid strands define preferred points.

Platonia is a strange place about which many mathematicians claim to visit regularly. I already proposed that the generalization of space-time point by bringing in the infinite number theoretical anatomy of real (and octonionic) units might allow to realize number theoretical Brahman=Atman identity by representing WCW in terms of the number theoretic anatomy of space-time points. This kind of representation would certainly be the most audacious idea that physical mathematician could dare to think of.

8.3.4 Is quantal Boolean reverse engineering possible?

The quantal version of Boolean algebra means that the basic logical functions have quantum inverses. The inverse of $C = A \wedge B$ represents the quantum superposition of all pairs A and B for which $A \wedge B = C$ hols true. Same is true for $\vee$. How could these additional quantum logical functions with no classical counterparts extend the capacities of logician?

What comes in mind is logical reverse engineering. Consider the standard problem solving situation repeatedly encountered by my hero Hercule Poirot. Someone has been murdered. Who could have done it? Who did it? Actually scientists who want to explain instead of just applying the method to get additional items to the CVC, meet this kind of problem repeatedly. One has something which looks like an experimental anomaly and one has to explain it. Is this anomaly genuine or is it due to a systematic error in the information processing? Could the interpretation of data be somehow wrong? Is the model behind experiments based on existing theory really correct or has something very delicate been neglected? If a genuine anomaly is in question (someone has been really murdered- this is always obvious in the tales about the deeds of Hercule Poirot since the mere presence of Hercule guarantees the murder unless it has been already done), one encounters what might be called Poirot problem in honor of my hero. As a matter fact, from the point of view of Boolean algebra, one has the same reverse Boolean engineering problem irrespective of whether it was a genuine anomaly or not.

This brings in my mind the enormously simplified problem. The logical statement C is found to be true. Which pairs A, B could have implied C as $C = A \wedge B$ (or $A \vee B$). Of course, much more complex situations can be considered where C corresponds to some logical function $C = f(A_1, A_2, ..., A_n)$. Quantum Poirot could use quantum computer able to realize the co-gates for gatesandOR (essentially time reversals) and write a quantum computer program solving the problem by constructing the Boolean co-function of Boolean function f.

What would happen in TGD Universe obeying zero energy ontology (ZEO) is following.

1. The statement C is represented as as positive energy part of zero energy state (analogous to initial state of physical event) and $A_1, ..A_n$ is represented as one state in the quantum superposition of final states representing various value combinations for $A_1, ..., A_n$. Zero energy states (rather than only their evolution) represents the arrow of time. The M-matrix characterizing time-like entanglement between positive and negative energy states generalizes generalizes S-matrix. S-matrix is such that initial states have well defined particle numbers

and other quantum numbers whereas final states do not. They are analogous to the outcomes of quantum measurement in particle physics.

2. Negentropy Maximization Principle [K8] maximizing the information contents of conscious experience (sic!) forces state function reduction to one particular $A_1, ..., A_n$ and one particular value combination consistent with C is found in each state function reduction. At the ensemble level one obtains probabilities for various outcomes and the most probable combination might represent the most plausible candidate for the murderer in quantum Poirot problem. Also in particle physics one can only speak about plausibility of the explanation and this leads to the endless n sigma talk. Note that it is absolutely essential that state function reduction occurs. Ironically, quantum problem solving causes dissipation at the level of ensemble but the ensemble probabilities carry actually information! Second law of thermodynamics tells us that Nature is a pathological problem solver- just like my hero!

3. In TGD framework basic logical binary operations have a representation at the level of Boolean algebra realized in terms of fermionic oscillator operators. They have also space-time correlates realized topologically. $\wedge$ has a representation as the analog of three-vertex of Feynman graph for partonic 2-surfaces: partonic 2-surfaces are glued along the ends to form outgoing partonic 2-surface. $\vee$ has a representation as the analog of stringy trouser vertex in which partonic surfaces fuse together. Here TGD differs from string models in a profound manner.

To conclude, I am a Boolean dilettante and know practically nothing about what quantum computer theorists have done- in particular I do not know whether they have considered quantum inverse gages. My feeling is that only the gates with bits replaced with qubits are considered: very natural when one thinks in terms of Boolean logic. If this is really the case, quantal co-AND and co-OR having no classical counterparts would bring a totally new aspect to quantum computation in solving problems in which one cannot do without (quantum) Poirot and his little gray (quantum) brain cells.

8.3.5 How to understand transcendental numbers in terms of infinite integers?

Santeri Satama made in my blog (see `http://tinyurl.com/yd9nh9fy`) a very interesting question about transcendental numbers. The reformulation of Santeri's question could be "How can one know that given number defined as a limit of rational number is genuinely algebraic or transcendental?". I answered to the question and since it inspired a long sequence of speculations during my morning walk on sands of Tullinniemi I decided to expand my hasty answer to a blog posting.

The basic outcome was the proposal that by bringing TGD based view about infinity based on infinite primes, integers, and rationals one could regard transcendental numbers as algebraic numbers by allowing genuinely infinite numbers in their definition.

1. In the definition of any transcendental as a limit of algebraic number (root of a polynomial and rational in special case) in which integer n approaches infinity one can replace n with any infinite integer. The transcendental would be an algebraic number in this generalized sense. Among other things this might allow polynomials with degree given by infinite integer if they have finite number of terms. Also mathematics would be generalized number theory, not only physics!

2. Each infinite integer would give a different variant of the transcendental: these variants would have different number theoretic anatomies but with respect to real norm they would be identical.

3. This would extend further the generalization of number concept obtained by allowing all infinite rationals which reduce to units in real sense and would further enrich the infinitely rich number theoretic anatomy of real point and also of space-time point. Space-time point would be the Platonia. One could call this number theoretic Brahman=Atman identity or algebraic holography.

1. How can one know that the real number is transcendental?

The difficulty of telling whether given real number defined as a limit of algebraic number boils down to the fact that there is no numerical method for telling whether this kind of number is rational, algebraic, or transcendental. This limitation of numerics would be also a restriction of cognition if p-adic view about it is correct. One can ask several questions. What about infinite-P p-adic numbers: if they make sense could it be possible to cognize also transcendentally? What can we conclude from the very fact that we cognize transcendentals? Transcendentality can be proven for some transcendentals such as π. How this is possible? What distinguishes "knowably transcendentals" like π and e from those, which are able to hide their real number theoretic identity?

1. Certainly for "knowably transcendentals" there must exist some process revealing their transcendental character. How π and e are proven to be transcendental? What in our mathematical cognition makes this possible? First of all one starts from the definitions of these numbers. e can be defined as the limit of the rational number $(1+1/n)^n$ and 2π could be defined as the limit for the length of the circumference of a regular n-side polygon and is a limit of an algebraic number since Pythagoras law is involved in calculating the length of the side. The process of proving "knowable transcendentality" would be a demonstration that these numbers cannot be solutions of any polynomial equation.

2. Squaring of circle is not possible because π is transcendental. When I search Wikipedia for squaring of circle (see `http://tinyurl.com/yaf6nf99`) I find a link to Weierstrass theorem (see `http://tinyurl.com/5y2gfr`) allowing to prove that π and e are transcendentals. In the formulation of Baker this theorem states the following: If α_1, ..., α_n are distinct algebraic numbers then the numbers e^{α_1}, ..., e^{α_n} are linearly independent over algebraic numbers and therefore transcendentals. One says that the extension $Q(e^{\alpha_1},...,e^{\alpha_n})$ of rationals has transcendence degree n over Q. This is something extremely deep and unfortunately I do not know what is the gist of the proof. In any case the proof defines a procedure of demonstrating "knowable transcendentality" for these numbers. The number of these transcendentals is huge but countable and therefore vanishingly small as compared to the uncountable cardinality of all transcendentals.

3. This theorem allows to prove that π and e are transcendentals. Suppose on the contrary that π is algebraic number. Then also $i\pi$ would be algebraic and the previous theorem would imply that $e^{i\pi} = -1$ is transcendental. This is of course a contradiction. Theorem also implies that e is transcendental. But how do we know that $e^{i\pi} = -1$ holds true? Euler deduced this from the connection between exponential and trigonometric functions understood in terms of complex analysis and related number theory. Clearly, rational functions and exponential function and its inverse -logarithm- continued to complex plane are crucial for defining e and π and proving also $e^{i\pi} = -1$. Exponent function and logarithm appear everywhere in mathematics: in group theory for instance. All these considerations suggest that "knowably transcendental" is a very special mathematical property and deserves a careful analysis.

2. Exponentiation and formation of set of subsets as transcendence

What is so special in exponentiation? Why it sends algebraic numbers to "knowably transcendentals". One could try to understand this in terms of exponentiation which for natural numbers has also an interpretation in terms of power set just as product has interpretation in terms of Cartesian product.

1. In Cantor's approach to the notion of infinite ordinals exponentiation is involved besides sum and product. All three binary operations - sum, product, exponent are expressed set theoretically. Product and sum are "algebraic" operations. Exponentiation is "non-algebraic" binary operation defined in terms of power set (set of subsets). For m and n defining the cardinalities of sets X and Y, m^n defines the cardinality of the set Y^X defining the number of functions assigning to each point of Y a point of X. When X is two-element set (bits 0 and 1) the power set is just the set of all subsets of Y which bit 1 assigned to the subset and 0 with its complement. If X has more than two elements one can speak of decompositions of Y to subsets colored with different colors- one color for each point of X.

2. The formation of the power set (or of its analog for the number of colors larger than 2) means going to the next level of abstraction: considering instead of set the set of subsets or studying the set of functions from the set. In the case of Boolean algebras this means formation of statements about statements. This could be regarded as the set theoretic view about transcendence.

3. What is interesting that 2-adic integers would label the elements of the power set of integers (all possible subsets would be allowed, for finite subsets one would obtain just natural numbers) and p-adic numbers the elements in the set formed by coloring integers with p colors. One could thus say that p-adic numbers correspond naturally to the notion of cognition based on power sets and their finite field generalizations.

4. But can one naively transcend the set theoretic exponent function for natural numbers to that defined in complex plane? Could the "knowably transcendental" property of numbers like e and π reduce to the transcendence in this set theoretic sense? It is difficult to tell since this notion of power applies only to integers m, n rather than to a pair of transcendentals e, π. Concretization of $e^{i\pi}$ in terms of sets seems impossible: it is very difficult to imagine what sets with cardinality e and π could be.

3. Infinite primes and transcendence

TGD suggests also a different identification of transcendence not expressible as formation of a power set or its generalizations.

1. The notion of infinite primes replaces the set theoretic notion of infinity with purely number theoretic one.

   (a) The mathematical motivation could be the need to avoid problems like Russell's antinomy. In Cantorian world a given ordinal is identified as the ordered set of all ordinals smaller than it and the set of all ordinals would define an ordinal larger than every ordinal and at the same time member of all ordinals.

   (b) The physical motivation for infinite primes is that their construction corresponds to a repeated second quantization of an arithmetic supersymmetric quantum field theory such that the many particle states of the previous level become elementary particles of the new level. At the lowest level finite primes label fermionic and bosonic states. Besides free many-particle states also bound states are obtained and correspond at the first level of the hierarchy to genuinely algebraic roots of irreducible polynomials.

   (c) The allowance of infinite rationals which as real numbers reduce to real units implies that the points of real axes have infinitely rich number theoretical anatomy. Space-time point would become the Platonia. One could speak of number theoretic Brahman=Atman identity or algebraic holography. The great vision is that the World of Classical Worlds has a mathematical representation in terms of the number theoretical anatomy of space-time point.

2. Transcendence in purely number theoretic sense could mean a transition to a higher level in the hierarchy of infinite primes. The scale of new infinity defined as the product of all prime at the previous level of hierarchy would be infinitely larger than the previous one. Quantization would correspond to abstraction and transcendence.

This idea inspires some questions.

1. Could infinite integers allow the reduction of transcendentals to algebraic numbers when understood in general enough sense. Could real algebraic numbers be reduced to infinite rationals with finite real values (for complex algebraic numbers this is certainly not the case)? If so, then all real numbers would be rationals identified as ratios of possibly infinite integers and having finite value as real numbers? This turns out to be too strong a statement. The statement that all real numbers can be represented as finite or infinite algebraic numbers looks however sensible and would reduce mathematics to generalized number theory by reducing limiting procedure involved with the transition from rationals to reals to algebraic transcendence. This applies also to p-adic numbers.

2. p-Adic cognition for finite values of prime p does not explain why we have the notions of π and e and more generally, that of transcendental number. Could the replacement of finite-p p-adic number fields with infinite-P p-adic number fields allow us to understand our own mathematical cognition? Could the infinite-P p-adic number fields or at least integers and corresponding space-time sheets make possible mathematical cognition able to deduce analytic formulas in which transcendentals and transcendental functions appear making it possible to leave the extremely restricted realm of numerics and enter the realm of mathematics? Lie group theory would represent a basic example of this transcendental aspect of cognition. Maybe this framework might allow to understand why we can have the notion of transcendental number!

4. Identification of real transcendentals as infinite algebraic numbers with finite value as real numbers

The following observations suggests that it could be possible to reduce transcendentals to generalized algebraic numbers in the framework provided by infinite primes. This would mean that not only physics but also mathematics (or at least "physical mathematics") could be seen as generalized number theory.

1. In the definition of any transcendental as an $n \to \infty$ limit of algebraic number (root of a polynomial and rational in special case), one can replace n with any infinite integer if n appears as an argument of a function having well defined value at this limit. If n appears as the number of summands or factors of product, the replacement does not make sense. For instance, an algebraic number could be defined as a limit of Taylor series by solving the polynomial equation defining it. The replacement of the upper limit of the series with infinite integer does not however make sense. Only transcendentals (and possibly also some algebraic numbers) allowing a representation as $n \to \infty$ limit with n appearing as argument of expression involving a finite number of terms can have representation as infinite algebraic number. The rule would be simple.

   Transcendentals or algebraic numbers allowing an identification as infinite algebraic number must correspond to a term of a sequence with a fixed number of terms rather than sum of series or infinite product.

2. Each infinite integer gives a different variant of the transcendental: these variants would have different number theoretic anatomies but with respect to the real norm they would be identical.

3. The heuristic guess is that any genuine algebraic number has an expression as Taylor series obtained by writing the solution of the polynomial equation as Tarylor expansion. If so, algebraic numbers must be introduced in the standard manner and do not allow a representation as infinite rationals. Only transcendentals would allow a representation as infinite rationals or infinite algebraic numbers. The infinite variety of representation in terms of infinite integers would enormously expand the number theoretical anatomy of the real point. Do all transcendentals allow an expression containing a finite number of terms and N appearing as argument? Or is this the defining property of only "knowably transcendentals" ?

One can consider some examples to illustrate the situation.

1. The transcendental π could be defined as $\pi_N = -iN(e^{i\pi/N} - 1)$, where $e^{i\pi/N}$ is N: th root of unity for infinite integer N and as a real number real unit. In real sense the limit however gives π. There are of course very many definitions of π as limits of algebraic numbers and each gives rise to infinite variety of number theoretic anatomies of π.

2. One can also consider the roots $exp(i2\pi n/N)$ of the algebraic equation $x^N = 1$ for infinite integer N. One might define the roots as limits of Taylor series for the exponent function but it does not make sense to define the limit when the cutoff for the Taylor series approaches some infinite integer. These roots would have similar multiplicative structure as finite roots of unity with p^n: th roots with p running over primes defining the generating roots. The presence of N^{th} roots of unity f or infinite N would further enrich the infinitely rich number theoretic anatomy of real point and therefore of space-time points.

3. There would be infinite variety of Neper numbers identified as $e_N = (1+1/N)^N$, N any infinite integer. Their number theoretic anatomies would be different but as real numbers they would be identical.

To conclude, the talk about infinite primes might sound weird in the ears of a layman but mathematicians do not lose their peace of mind when they here the word "infinity". The notion of infinity is relative. For instance, infinite integers are completely finite in p-adic sense. One can also imagine completely "real-worldish" realizations for infinite integers (say as states of repeatedly second quantized arithmetic quantum field theory and this realization might provide completely new insights about how to understand bound states in ordinary QFT).

9 Local Zeta Functions, Galois Groups, And Infinite Primes

The recent view about TGD leads to some conjectures about Riemann Zeta.

1. Non-trivial zeros should be algebraic numbers.
2. The building blocks in the product decomposition of ζ should be algebraic numbers for non-trivial zeros of zeta.
3. The values of zeta for their combinations with positive imaginary part with positive integer coefficients should be algebraic numbers.

These conjectures are motivated by the findings that Riemann Zeta seems to be associated with critical systems and by the fact that non-trivial zeros of zeta are analogous to complex conformal weights. The necessity to make such a strong conjectures, in particular conjecture c), is an unsatisfactory feature of the theory and one could ask how to modify this picture. Also a clear physical interpretation of Riemann zeta is lacking.

9.1 Zeta Function And Infinite Primes

Fermionic Zeta function is expressible as a product of fermionic partition functions $Z_{F,p} = 1 + p^{-z}$ and could be seen as an image of X under algebraic homomorphism mapping prime p to $Z_{F,p}$ defining an analog of prime in the commutative function algebra of complex numbers. For hyper-octonionic infinite primes the contribution of each p to the norm of X is same finite power of p since only single representative from each Lorentz equivalence class is included, and there is one-one correspondence with ordinary primes so that an appropriate power of ordinary ζ_F might be regarded as a representation of X also in the case of hyper-octonionic primes.

Infinite primes suggest a generalization of the notion of ζ function. Real-rational infinite prime $X \pm 1$ would correspond to $\zeta_F \pm 1$. General infinite prime is mapped to a generalized zeta function by dividing ζ_F with the product of partition functions $Z_{F,p}$ corresponding to fermions kicked out from sea. The same product multiplies "1". The powers p^n present in either factor correspond to the presence of n bosons in mode p and to a factor $Z^n_{p,B}$ in corresponding summand of the generalized Zeta.

For zeros of ζ_F which are same as those of Riemann ζ the image of infinite part of infinite prime vanishes and only the finite part is represented faithfully. Whether this might have some physical meaning is an interesting question.

9.2 Local Zeta Functions And Weil Conjectures

Riemann Zeta is not the only zeta [A1, A9]. There is entire zoo of zeta functions and the natural question is whether some other zeta sharing the basic properties of Riemann zeta having zeros at critical line could be more appropriate in TGD framework.

The so called local zeta functions analogous to the factors $\zeta_p(s) = 1/(1-p^{-s})$ of Riemann Zeta can be used to code algebraic data about say numbers about solutions of algebraic equations reduced to finite fields. The local zeta functions appearing in Weil's conjectures [A8] associated with finite fields $G(p,k)$ and thus to single prime. The extensions $G(p,nk)$ of this finite field are considered. These local zeta functions code the number for the points of algebraic variety for given value of n. Weil's conjectures also state that if X is a mod p reduction of non-singular complex projective variety then the degree for the polynomial multiplying the product $\zeta(s)\times\zeta(s-1)$ equals to Betti number. Betti number is 2 times genus in 2-D case.

It has been proven that the zetas of Weil are associated with single prime p, they satisfy functional equation, their zeros are at critical lines, and rather remarkably, they are rational functions of p^{-s}. For instance, for elliptic curves zeros are at critical line [A8].

The general form for the local zeta is $\zeta(s) = exp(G(s))$, where $G = \sum g_n p^{-ns}$, $g_n = N_n/n$, codes for the numbers N_n of points of algebraic variety for n^{th} extension of finite field F with nk elements assuming that F has $k = p^r$ elements. This transformation resembles the relationship $Z = exp(F)$ between partition function and free energy $Z = exp(F)$ in thermodynamics.

The exponential form is motivated by the possibility to factorize the zeta function into a product of zeta functions. Note also that in the situation when N_n approaches constant N_∞, the division of N_n by n gives essentially $1/(1-N_\infty p^{-s})$ and one obtains the factor of Riemann Zeta at a shifted argument $s - log_p(N_\infty)$. The local zeta associated with Riemann Zeta corresponds to $N_n = 1$.

9.3 Galois Groups, Jones Inclusions, And Infinite Primes

Langlands program [A4, A11] is an attempt to unify mathematics using the idea that all zeta functions and corresponding theta functions could emerge as automorphic functions giving rise to finite-dimensional representations for Galois groups (Galois group is defined as a group of automorphisms of the extension of field F leaving invariant the elements of F). The basic example corresponds to rationals and their extensions. Finite fields $G(p,k)$ and their extensions $G(p,nk)$ represents another example. The largest extension of rationals corresponds to algebraic numbers (algebraically closed set). Although this non-Abelian group is huge and does not exist in the usual sense of the word its finite-dimensional representations in groups $GL(n,Z)$ make sense.

For instance, Edward Witten is working with the idea that geometric variant of Langlands duality could correspond to the dualities discovered in string model framework and be understood in terms of topological version of four-dimensional $N = 4$ super-symmetric YM theory [A12]. In particular, Witten assigns surface operators to the 2-D surfaces of 4-D space-time. This brings unavoidably in mind partonic 2-surfaces and TGD as $N = 4$ super-conformal almost topological QFT.

This observation stimulates some ideas about the role of zeta functions in TGD if one takes the vision about physics as a generalized number theory seriously.

9.3.1 Galois groups, Jones inclusions, and quantum measurement theory

The Galois representations appearing in Langlands program could have a concrete physical/cognitive meaning.

1. The Galois groups associated with the extensions of rationals have a natural action on partonic 2-surfaces represented by algebraic equations. Their action would reduce to permutations of roots of the polynomial equations defining the points with a fixed projection to the above mentioned geodesic sphere S^2 of CP_2 or δM^4_+. This makes possible to define modes of induced spinor fields transforming under representations of Galois groups. Galois groups would also have a natural action on WCW -spinor fields. One can also speak about WCW spinor s invariant under Galois group.

2. Galois groups could be assigned to Jones inclusions having an interpretation in terms of a finite measurement resolution in the sense that the discrete group defining the inclusion leaves invariant the operators generating excitations which are not detectable.

3. The physical interpretation of the finite resolution represented by Galois group would be based on the analogy with particle physics. The field extension K/F implies that the primes (more precisely, prime ideals) of F decompose into products of primes (prime ideals) of K. Physically this corresponds to the decomposition of particle into more elementary constituents, say hadrons into quarks in the improved resolution implied by the extension $F \to K$. The interpretation in terms of cognitive resolution would be that the primes associated with the higher extensions of rationals are not cognizable: in other words, the

observed states are singlets under corresponding Galois groups: one has algebraic/cognitive counterpart of color confinement.

4. For instance, the system labeled by an ordinary p-adic prime could decompose to a system which is a composite of Gaussian primes. Interestingly, the p-adic length scale range 10 nm-5 μm contains as many as four scaled up electron Compton lengths assignable to Gaussian Mersennes $M_k = (1+i)^k - 1$, $k = 151, 157, 163, 167$, which suggests that the emergence of living matter means an improved cognitive resolution.

9.3.2 Galois groups and infinite primes

In particular, the notion of infinite prime suggests a manner to realize the modular functions as representations of Galois groups. Infinite primes might also provide a new perspective to the concrete realization of Langlands program.

1. The discrete Galois groups associated with various extensions of rationals and involved with modular functions which are in one-one correspondence with zeta functions via Mellin transform defined as $\sum x_n n^{-s} \to \sum x_n z^n$ [A5]. Various Galois groups would have a natural action in the space of infinite primes having interpretation as Fock states and more general bound states of an arithmetic quantum field theory.

2. The number theoretic anatomy of space-time points due to the possibility to define infinite number of number theoretically non-equivalent real units using infinite rationals [L1] allows the imbedding space points themselves to code holographically various things. Galois groups would have a natural action in the space of real units and thus on the number theoretical anatomy of a point of imbedding space.

3. Since the repeated second quantization of the super-symmetric arithmetic quantum field theory defined by infinite primes gives rise to a huge space of quantum states, the conjecture that the number theoretic anatomy of imbedding space point allows to represent WCW (the world of classical worlds associated with the light-cone of a given point of H) and WCW spinor fields emerges naturally [L1].

4. Since Galois groups G are associated with inclusions of number fields to their extensions, this inclusion could correspond at quantum level to a generalized Jones inclusion $\mathcal{N} \subset \mathcal{M}$ such that G acts as automorphisms of $\mathcal{M}$ and leaves invariant the elements of $\mathcal{N}$. This might be possible if one allows the replacement of complex numbers as coefficient fields of hyper-finite factors of type II_1 with various algebraic extensions of rationals. Quantum measurement theory with a finite measurement resolution defined by Jones inclusion $\mathcal{N} \subset \mathcal{M}$ [L2] could thus have also a purely number theoretic meaning provided it is possible to define a non-trivial action of various Galois groups on WCW spinor fields via the imbedding of the configuration space spinors to the space of infinite integers and rationals (analogous to the imbedding of space-time surface to imbedding space).

This picture allows to develop rather fascinating ideas about mathematical structures and their relationship to physical world. For instance, the functional form of a map between two sets the points of the domain and target rather than only its value could be coded in a holographic manner by using the number theoretic anatomy of the points. Modular functions giving rise to generalized zeta functions would emerge in especially natural manner in this framework. WCW spinor fields would allow a physical realization of the holographic representations of various maps as quantum states.

9.4 Prime Hilbert Spaces And Infinite Primes

There is a result of quantum information science providing an additional reason why for p-adic physics. Suppose that one has N-dimensional Hilbert space which allows $N+1$ unbiased basis. This means that the moduli squared for the inner product of any two states belonging to different basis equals to $1/N$. If one knows all transition amplitudes from a given state to all states of all $N+1$ mutually unbiased basis, one can fully reconstruct the state. For $N = p^n$ dimensional $N+1$

unbiased basis can be found and the article of Durt [A18] gives an explicit construction of these basis by applying the properties of finite fields. Thus state spaces with p^n elements - which indeed emerge naturally in p-adic framework - would be optimal for quantum tomography. For instance, the discretization of one-dimensional line with length of p^n units would give rise to p^n-dimensional Hilbert space of wave functions.

The observation motivates the introduction of prime Hilbert space as as a Hilbert space possessing dimension which is prime and it would seem that this kind of number theoretical structure for the category of Hilbert spaces is natural from the point of view of quantum information theory. One might ask whether the tensor product of mutually unbiased bases in the general case could be constructed as a tensor product for the bases for prime power factors. This can be done but since the bases cannot have common elements the number of unbiased basis obtained in this manner is equal to $M+1$, where M is the smallest prime power factor of N. It is not known whether additional unbiased bases exists.

9.4.1 Hierarchy of prime Hilbert spaces characterized by infinite primes

The notion of prime Hilbert space provides also a new interpretation for infinite primes, which are in 1-1 correspondence with the states of a supersymmetric arithmetic QFT. The earlier interpretation was that the hierarchy of infinite primes corresponds to a hierarchy of quantum states. Infinite primes could also label a hierarchy of infinite-D prime Hilbert spaces with product and sum for infinite primes representing unfaithfully tensor product and direct sum.

1. At the lowest level of hierarchy one could interpret infinite primes as homomorphisms of Hilbert spaces to generalized integers (tensor product and direct sum mapped to product and sum) obtained as direct sum of infinite-D Hilbert space and finite-D Hilbert space. (In)finite-D Hilbert space is (in)finite tensor product of prime power factors. The map of N-dimensional Hilbert space to the set of N-orthogonal states resulting in state function reduction maps it to N-element set and integer N. Hence one can interpret the homomorphism as giving rise to a kind of shadow on the wall of Plato's cave projecting (shadow quite literally!) the Hilbert space to generalized integer representing the shadow. In category theoretical setting one could perhaps see generalize integers as shadows of the hierarchy of Hilbert spaces.

2. The interpretation as a decomposition of the universe to a subsystem plus environment does not seem to work since in this case one would have tensor product. Perhaps the decomposition could be to degrees of freedom to those which are above and below measurement resolution. One could of course consider decomposition to a tensor product of bosonic and fermionic state spaces.

3. The construction of the Hilbert spaces would reduce to that of infinite primes. The analog of the fermionic sea would be infinite-D Hilbert space which is tensor product of all prime Hilbert spaces H_p with given prime factor appearing only once in the tensor product. One can "add n bosons" to this state by replacing of any tensor factor H_p with its n+1: th tensor power. One can "add fermions" to this state by deleting some prime factors H_p from the tensor product and adding their tensor product as a finite-direct summand. One can also "add n bosons" to this factor.

4. At the next level of hierarchy one would form infinite tensor product of all infinite-dimensional prime Hilbert spaces obtained in this manner and repeat the construction. This can be continued ad infinitum and the construction corresponds to abstraction hierarchy or a hierarchy of statements about statements or a hierarchy of n: th order logics. Or a hierarchy of space-time sheets of many-sheeted space-time. Or a hierarchy of particles in which certain many-particle states at the previous level of hierarchy become particles at the new level (bosons and fermions). There are many interpretations.

5. Note that at the lowest level this construction can be applies also to Riemann Zeta function. ζ would represent fermionic vacuum and the addition of fermions would correspond to a removal of a product of corresponding factors ζ_p from ζ and addition of them to the resulting truncated ζ function. The addition of bosons would correspond to multiplication by a power of appropriate ζ_p. The analog of ζ function at the next level of hierarchy would be product

of all these modified ζ functions and might well fail to exist since the product might typically converge to either zero or infinity.

9.4.2 Hilbert spaces assignable to infinite integers and rationals make also sense

1. Also infinite integers make sense since one can form tensor products and direct sums of infinite primes and of corresponding Hilbert spaces. Also infinite rationals exist and this raises the question what kind of state spaces inverses of infinite integers mean.

2. Zero energy ontology suggests that infinite integers correspond to positive energy states and their inverses to negative energy states. Zero energy states would be always infinite rationals with real norm which equals to real unit.

3. The existence of these units would give for a given real number an infinite rich number theoretic anatomy so that single space-time point might be able to represent quantum states of the entire universe in its anatomy (number theoretical Brahman=Atman). Also the world of classical worlds (light-like 3-surfaces of the imbedding space) might be imbeddable to this anatomy so that basically one would have just space-time surfaces in 8-D space and WCW would have representation in terms of space-time based on generalized notion of number. Note that infinitesimals around a given number would be replaced with infinite number of number-theoretically non-equivalent real units multiplying it.

9.4.3 Should one generalize the notion of von Neumann algebra?

Especially interesting are the implications of the notion of prime Hilbert space concerning the notion of von Neumann algebra -in particular the notion of hyper-finite factors of type II_1 playing a key role in TGD framework. Does the prime decomposition bring in additional structure? Hyper-finite factors of type II_1 are canonically represented as infinite tensor power of 2×2 matrix algebra having a representation as infinite-dimensional fermionic Fock oscillator algebra and allowing a natural interpretation in terms of spinors for the world of classical worlds having a representation as infinite-dimensional fermionic Fock space.

Infinite primes would correspond to something different: a tensor product of all $p\times p$ matrix algebras from which some factors are deleted and added back as direct summands. Besides this some factors are replaced with their tensor powers. Should one refine the notion of von Neumann algebra so that one can distinguish between these algebras as physically non-equivalent? Is the full algebra tensor product of this kind of generalized hyper-finite factor and hyper-finite factor of type II_1 corresponding to the vibrational degrees of freedom of 3-surface and fermionic degrees of freedom? Could p-adic length scale hypothesis - stating that the physically favored primes are near powers of 2 - relate somehow to the naturality of the inclusions of generalized von Neumann algebras to HFF of type II_1?

10 Miscellaneous

This section is devoted to what might be called miscellaneous since it does not relate directly to quantum TGD.

10.1 The Generalization Of The Notion Of Ordinary Number Field

The notion of infinite rationals leads also to the generalization of the notion of a finite number. The obvious generalization would be based on the allowance of infinitesimals. Much more interesting approach is however based on the observation that one obtains infinite number of real units by taking two infinite primes with a finite rational valued ratio q and by dividing this ratio by ordinary rational number q. As a real number the resulting number differs in no manner from ordinary unit of real numbers but in p-adic sense the points are not equivalent. This construction generalizes also to quaternionic and octonionic case.

Space-time points would become structured since infinite rationals normed to unity define naturally a gigantically infinite-dimensional free algebra generated by the units serving in well-define sense as Mother of All Algebras. The units of the algebra multiplying ordinary rational numbers (and also other elements) of various number fields are invisible at the level of real physics so that the interpretation as the space-time correlate of mathematical cognition realizing the idea of monad is natural. Universe would be an algebraic hologram with single point being able to represent the state of the Universe in its structure. Infinite rationals would allow the realization of the Platonia of all imaginable mathematical constructs at the level of space-time.

10.1.1 The generalized units for quaternions and octonions

In the case of real and complex rationals the group of generalized units generated by primes *resp.* infinite Gaussian primes is commutative. In the case of unit quaternions and hyper-quaternions group becomes non-commutative and in case of unit hyper-octonions the group is replaced by a kind non-associative generalization of group.

1. For infinite primes for which only finite number of bosonic and fermionic modes are excited it is possible to tell how the products AB and BA of two infinite primes explicitly since finite hyper-octonionic primes can be assumed to multiply the infinite integer X from say left.

2. Situation changes if an infinite number of bosonic excitations are present since one would be forced to move finite H- or O-primes past a infinite number of primes in the product AB. Hence one must simply assume that the group G generated by infinite units with infinitely many bosonic excitations is a free group. Free group interpretation means that non-associativity is safely localized inside infinite primes and reduced to the non-associativity of ordinary hyper-octonions. Needless to say free group is the best one can hope of achieving since free group allows maximal number of factor groups.

The free group G can be extended into a free algebra A by simply allowing superpositions of units with coefficients which are real-rationals or possibly complex rationals. Again free algebra fulfils the dreams as system with a maximal representative power. The analogy with quantum states defined as functions in the group is highly intriguing and unit normalization would correspond to the ordinary normalization of Schrödinger amplitudes. Obviously this would mean that single point is able to mimic quantum physics in its structure. Could state function reduction and preparation be represented at the level of space-time surfaces so that initial and final 3-surfaces would represent pure states containing only bound state entanglement or negentropic entanglement represented algebraically, and could the infinite rationals generating the group of quaternionic units (no sums over them) represent pure states?

The free algebra structure of A together with the absolutely gigantic infinite-dimensionality of the endless hierarchy of infinite rational units suggests that the resulting free algebra structure is universal in the sense that any algebra defined with coefficients in the field of rationals can be imbedded to the resulting algebra or represented as a factor algebra obtained by the sequence $A \rightarrow A_1 = A/I_1 \rightarrow A_1/I_2 ...$ where the ideal I_k is defined by $k:th$ relation in A_{k-1}.

Physically the embedding would mean that some field quantities defined in the algebra are restricted to the subalgebra. The representation of algebra B as an iterated factor algebra would mean that some field quantities defined in the algebra are constant inside the ideals I_k of A defined by the relations. For instance, the induced spinor field at space-time surface could have the same value for all points of A which differ by an element of the ideal. At WCW level, the WCW spinor field would be constant inside an ideal associated with the algebra of A-valued functions at space-time surfaces.

The units can be interpreted as defining an extension of rationals in C, H, or O. Galois group is defined as automorphisms of the extension mapping the original number field to itself and obviously the transformations $x \rightarrow gxg^{-1}$, where g belongs to the extended number field act as automorphisms. One can regard also the extension by real units as the extended number field and in this case the automorphisms contain also the automorphisms induced by the multiplication of each infinite prime Π_i by a real unit U_i: $\Pi_i \rightarrow \hat{\Pi}_i = U_i \Pi_i$.

10.1.2 The free algebra generated by generalized units and mathematical cognition

One of the deepest questions in theory of consciousness concerns about the space-time correlates of mathematical cognition. Mathematician can imagine endlessly different mathematical structures.

Platonist would say that in some sense these structures exist. The claim classical physical worlds correspond to certain 4-surfaces in $M^4_+ \times CP_2$ would leave out all these beautiful mathematical structures unless they have some other realization than the physical one.

The free algebra A generated by the generalized multiplicative units of rationals allows to understand how Platonia is realized at the space-time level. A has no correlate at the level of real physics since the generalized units correspond to real numbers equal to one. This holds true also in quaternionic and octonionic cases since one can require that the units have net quaternionic and octonionic phases equal to one. By its gigantic size A and free algebra character might be able represent all possible algebras in the proposed manner. Also non-associative algebras can be represented.

Algebraic equations are the basic structural building blocks of mathematical thinking. Consider as a simple example the equation $AB = C$. The equations are much more than tautologies since they contain the information at the left hand side about the variables of the algebraic operation giving the outcome on the right hand side. For instance, in the case of multiplication $AB = C$ the information about the factors is present although it is completely lost when the product is evaluated. These equations pop up into our consciousness in some mysterious manner and the question is what are the space-time correlates of these experiences suggested to exist by quantum-classical correspondence.

The algebra of units is an excellent candidate for the sought for correlate of mathematical cognition. Leibniz might have been right about his monads! The idealization is in complete accordance with the idea about the Universe as an algebraic hologram taken to its extreme. One might perhaps say that each point represents an equation.

One could also try interpret generalized Feynman diagrams as sequences of mathematical operations. For instance, the scattering $AB \to CD$ by exchange of particle C could be seen as an arithmetic operation $AB \to (AE^{-1})(EB) = CD$. If this is really the case, then at least tree diagrams might allow interpretation in terms of arithmetic operations for the complexified octonionic units. In case of loop diagrams it seems that one must allow sums over units.

10.1.3 When two points are cobordant?

Topological quantum field theories have led to a dramatic success in the understanding of 3- and 4-dimensional topologies and cobordisms of these manifolds (two n-manifolds are cobordant if there exists an $n+1$-manifold having them as boundaries). In his thought-provoking and highly inspiring article Pierre Cartier [A14] poses a question which at first sounds absurd. What might be the counterpart of cobordism for points? The question is indeed absurd unless the points have some structure.

If one takes seriously the idea that each point of space-time sheet corresponds to a unit defined by an infinite rational, the obvious question is under what conditions there is a continuous line connecting these points with continuity being defined in some generalized sense. In real sense the line is continuous always but in p-adic sense only if all p-adic norms of the two units are identical. Since the p-adic norm of the unit of $Y(n/m) = X/\Pi(n/m)$ is that of $q = n/m$, the norm of two infinite rational numbers is same only if they correspond to the same ordinary rational number.

Suppose that one has

$$
\begin{aligned}
Y_I &= \frac{\prod_i Y(q^I_{1i})}{\prod_i Y(q^I_{2i})} \ , \quad Y_F = \frac{\prod_i Y(q^F_{1i})}{\prod_i Y(q^F_{2i})} \ , \\
q^I_{ki} &= \frac{n^I_{k_i}}{m^I_{k_i}} \ , \qquad q^F_{ki} = \frac{n^F_{k_i}}{m^F_{k_i}} \ ,
\end{aligned} \tag{10.1}
$$

Here $m^{\cdot}_{.}$ representing arithmetic many-fermion state is a square free integer and $n^{\cdot}_{.}$ representing arithmetic many-boson state is an integer having no common factors with $m^{\cdot}_{.}$.

The two units have same p-adic norm in all p-adic number fields if the rational numbers associated with Y_I and Y_F are same:

$$
\frac{\prod_i q^I_{1i}}{\prod_i q^I_{2i}} = \frac{\prod_i q^F_{1i}}{\prod_i q^F_{2i}} \ . \tag{10.2}
$$

The logarithm of this condition gives a conservation law of energy encountered in arithmetic quantum field theories, where the energy of state labeled by the prime p is $E_p = log(p)$:

$$\begin{aligned} E^I &= \sum_i log(n^I_{1i}) - \sum_i log(n^I_{2i}) - \sum_i log(m^I_{1i}) + \sum_i log(m^I_{2i}) = \\ &= \sum_i log(n^F_{1i}) - \sum_i log(n^F_{2i}) - \sum_i log(m^F_{1i}) + \sum_i log(m^F_{2i}) = E^F \ . \end{aligned} \tag{10.3}$$

There are both positive and negative energy particles present in the system. The possibility of negative energies is indeed one of the basic predictions of quantum TGD distinguishing it from standard physics. As one might have expected, Y^I and Y^F represent the initial and final states of a particle reaction and the line connecting the two points represents time evolution giving rise to the particle reaction. In principle one can even localize various steps of the reaction along the line and different lines give different sequences of reaction steps but same overall reaction. This symmetry is highly analogous to the conformal invariance implying that integral in complex plane depends only on the end points of the curve.

Whether the entire four-surface should correspond to the same value of topological energy or whether E can be discontinuous at elementary particle horizons separating space-time sheets and represented by light-like 3-surfaces around wormhole contacts remains an open question. Discontinuity through elementary particle horizons would make possible the arithmetic analogs of poles and cuts of analytic functions since the limiting values of Y from different sides of the horizon are different. Note that the construction generalizes to the quaternionic and octonionic case.

10.1.4 TGD inspired analog for d-algebras

Maxim Kontsevich has done deep work with quantizations interpreted as a deformation of algebraic structures and there are deep connections with this work and braid group [A13]. In particular, the Grothendienck-Teichmueller algebra believed to act as automorphisms for the deformation structures acts as automorphisms of the braid group at the limit of infinite number of strands. I must admit that my miserable skills in algebra do not allow to go to the horrendous technicalities but occasionally I have the feeling that I have understood some general ideas related to this work. In his article "Operads and Motives in Deformation Quantization" Kontsevich introduces the notions of operad and d-algebras over operad. Without going to technicalities one can very roughly say that d-algebra is essentially d-dimensional algebraic structure, and that the basic conjecture of Deligne generalized and proved by Kontsevich states in its generalized form that $d + 1$-algebras have a natural action in all d-algebras.

In the proposed extension of various rationals a notion resembling that of universal d-algebra to some degree but not equivalent with it emerges naturally. The basic idea is simple.

1. Points correspond to the elements of the assumed to be universal algebra A which in this sense deserves the attribute $d = 0$ algebra. By its universality A should be able to represent any algebra and in this sense it cannot correspond $d = 0$-algebra of Kontsevich defined as a complex, that is a direct sum of vector spaces V_n and possessing d operation $V_n \rightarrow V_{n+1}$, satisfying $d^2 = 0$. Each point of a manifold represents one particular element of 0-algebra and one could loosely say that multiplication of points represents algebraic multiplication. This algebra has various subalgebras, in particular those corresponding to reals, complex numbers and quaternions. One can say that sub-algebra is non-associative, non-commutative, etc.. if its real evaluation has this property.

2. Lines correspond to evolutions for the elements of A which are continuous with respect to real (trivially) and all p-adic number fields. The latter condition is nontrivial and allows to interpret evolution as an evolution conserving number theoretical analog of total energy. Universal 1-group would consist of curves along which one has the analog of group valued field (group being the group of generalized units) having values in the universal 0-group G. The action of the 1-group in 0-group would simply map the element of 0-group at the first end of the curve its value at the second end. Curves define a monoid in an obvious manner.

The interpretation as a map to A allows pointwise multiplication of these mappings which generalizes to all values of d.

One could also consider the generalization of local gauge field so that there would be gauge potential defined in the algebra of units having values on A. This potential would define holonomy group acting on 0-algebra and mapping the element at the first end of the curve to its gauge transformed variant at the second end. In this case also closed curves would define non-trivial elements of the holonomy group. In fact, practically everything is possible since probably any algebra can be represented in the algebra generated by units.

3. Two-dimensional structures correspond to dynamical evolutions of one-dimensional structures. The simplest situation corresponds to 2-cubes with the lines corresponding to the initial and final values of the second coordinate representing initial and final states. One can also consider the possibility that the two-surface is topologically non-trivial containing handles and perhaps even holes. One could interpret this cognitive evolution as a 1-dimensional flow so that the initial points travel to final points. Obviously there is symmetry breaking involved since the second coordinate is in the role of time and this defines kind of time orientation for the surface.

4. The generalization to 4- and higher dimensional cases is obvious. One just uses d-manifolds with edges and uses their time evolution to define $d+1$-manifolds with edges. Universal 3-algebra is especially interesting from the point of view of braid groups and in this case the maps between initial and final elements of 2-algebra could be interpreted as braid operations if the paths of the elements along 3-surface are entangled. For instance field lines of Kähler gauge potential or of magnetic field could define this kind of braiding.

5. The d-evolutions define a monoid since one can glue two d-evolutions together if the outcome of the first evolution equals to the initial state of the second evolution. $d+1$-algebra also acts naturally in d-algebra in the sense that the time evolution $f(A \to B)$ assigns to the d-algebra valued initial state A a d-algebra valued final state and one can define the multiplication as $f(A \to B)C = B$ for $A = C$, otherwise the action gives zero. If time evolutions correspond to standard cubes one gets more interesting structure in this manner since the cubes differing by time translation can be identified and the product is always non-vanishing.

6. It should be possible to define generalizations of homotopy groups to what might be called "cognitive" homotopy groups. Effectively the target manifold would be replaced by the tensor product of an ordinary manifold and some algebraic structure represented in A. All kinds of "cognitive" homotopy groups would result when the image is cognitively non-contractible. Also homology groups could be defined by generalizing singular complex consisting of cubes with cubes having the hierarchical decomposition into time evolutions of time evolutions of... in some sub-algebraic structure of A. If one restricts time evolutions to sub-algebraic structures one obtains all kinds of homologies. For instance, associativity reduces 3-evolutions to paths in rational $SU(3)$ and since $SU(3)$ just like any Lie group has non-trivial 3-homology, one obtains nontrivial "cognitive" homology for 3-surfaces with non-trivial 3-homology.

The following heuristic arguments are inspired by the proposed vision about algebraic cognition and the conjecture that Grothendienck-Teichmueller group acts as automorphisms of Feynman diagrammatics relating equivalent quantum field theories to each other.

1. The operations of $d+1$-algebra realized as time evolution of d-algebra elements suggests an interpretation as cognitive counterparts for sequences of algebraic manipulations in d-algebra which themselves become elements of $d+1$ algebra. At the level of paths of points the sequences of algebraic operations correspond to transitions in which the number of infinite primes defining an infinite rational can change in discrete steps but is subject to the topological energy conservation guaranteeing the p-adic continuity of the process for all primes. Different paths connecting a and b represent different but equivalent manipulations sequences.

   For instance, at $d = 2$ level one has a pile of these processes and this in principle makes it possible an abstraction to algebraic rules involved with the process by a pile of examples. Higher values of d in turn make possible further abstractions bringing in additional parameters to the system. All kinds of algebraic processes can be represented in this manner. For instance, multiplication table can be represented as paths assigning to an the initial state product of elements a and b represented as infinite rationals and to the final state their product ab represented as single infinite rational. Representation is of course always approximate unless the algebra is finite. All kinds abstract rules such as various commutative diagrams, division of algebra by ideal by choosing one representative from each equivalence class of A/I as end point of the path, etc... can be represented in this manner.

2. There is also second manner to represent algebraic rules. Entanglement is a purely algebraic notion and it is possible to entangle the many-particle states formed as products of infinite rationals representing inputs of an algebraic operation A with the outcomes of A represented in the same manner such that the entanglement is consistent with the rule.

3. There is nice analogy between Feynman diagrams and sequences of algebraic manipulations. Multiplication ab corresponds to a map $A \otimes A \rightarrow A$ is analogous to a fusion of elementary particles since the product indeed conserves the number theoretical energy. Co-algebra operations are time reversals of algebra operations in this evolution. Co-multiplication Δ assigns to $a \in A$ an element in $A \otimes A$ via algebra homomorphism and corresponds to a decay of initial state particle to two final state particles. It defines co-multiplication assign to $a \otimes b \in A \otimes A$ an element of $A \otimes A \rightarrow A \otimes A \otimes A$ and corresponds to a scattering of elementary particles with the emission of a third particle. Hence a sequence of algebraic manipulations is like a Feynman diagram involving both multiplications and co-multiplications and thus containing also loops. When particle creation and annihilation are absent, particle number is conserved and the process represents algebra endomorphism $A \rightarrow A$. Otherwise a more general operation is in question. This analogy inspires the question whether particle reactions could serve as a blood and flesh representation for $d = 4$ algebras.

4. The dimension $d = 4$ is maximal dimension of single space-time evolution representing an algebraic operation (unless one allows the possibility that space-time and imbedding space dimensions are come as multiples of four and 8). Higher dimensions can be effectively achieved only if several space-time sheets are used defining $4n$-dimensional WCW . This could reflect some deep fact about algebras in general and also relate to the fact that 3- and 4-dimensional manifolds are the most interesting ones topologically.

10.2 One Element Field, Quantum Measurement Theory And ItsQ-Variant, And The Galois Fields Associated With Infinite Primes

John Baez talked in This Weeks Finds (Week 259) [B1] about one-element field - a notion inspired by the $q = exp(i2\pi/n) \rightarrow 1$ limit for quantum groups. This limit suggests that the notion of one-element field for which 0=1 - a kind of mathematical phantom for which multiplication and sum should be identical operations - could make sense. Physicist might not be attracted by this kind of identification.

In the following I want to articulate some comments from the point of view of quantum measurement theory and its generalization to q-measurement theory which I proposed for some years ago and which is represented above.

I also consider and alternative interpretation in terms of Galois fields assignable to infinite primes which form an infinite hierarchy. These Galois fields have infinite number of elements but the map to the real world effectively reduces the number of elements to 2: 0 and 1 remain different.

10.2.1 $q \rightarrow 1$ limit as transition from quantum physics to effectively classical physics?

The $q \rightarrow 1$ limit of quantum groups at q-integers become ordinary integers and n-D vector spaces reduce to n-element sets. For quantum logic the reduction would mean that 2^N-D spinor space becomes 2^N-element set. N qubits are replaced with N bits. This brings in mind what happens in the transition from wave mechanism to classical mechanics. This might relate in interesting manner to quantum measurement theory.

Strictly speaking, $q \rightarrow 1$ limit corresponds to the limit $q = exp(i2\pi/n)$, $n \rightarrow \infty$ since only roots of unity are considered. This also correspond to Jones inclusions at the limit when the discrete group Z_n or or its extension-both subgroups of $SO(3)$- to contain reflection has infinite elements. Therefore this limit where field with one element appears might have concrete physical meaning. Does the system at this limit behave very classically?

In TGD framework this limit can correspond to either infinite or vanishing Planck constant depending on whether one consider orbifolds or coverings. For the vanishing Planck constant one should have classicality: at least naively! In perturbative gauge theory higher order corrections come as powers of $g^2/4\pi\hbar$ so that also these corrections vanish and one has same predictions as given by classical field theory.

10.2.2 Q-measurement theory and $q \rightarrow 1$ limit

Q-measurement theory differs from quantum measurement theory in that the coordinates of the state space, say spinor space, are non-commuting. Consider in the sequel q-spinors for simplicity.

Since the components of quantum spinor do not commute, one cannot perform state function reduction. One can however measure the modulus squared of both spinor components which indeed commute as operators and have interpretation as probabilities for spin up or down. They have a universal spectrum of eigen values. The interpretation would be in terms of fuzzy probabilities and finite measurement resolution but may be in different sense as in case of HFF: s. Probability would become the observable instead of spin for q not equal to 1.

At $q \rightarrow 1$ limit quantum measurement becomes possible in the standard sense of the word and one obtains spin down or up. This in turn means that the projective ray representing quantum states is replaced with one of n possible projective rays defining the points of n-element set. For HFF: s of type II_1 it would be N-rays which would become points, N the included algebra. One might also say that state function reduction is forced by this mapping to *single* object at $q \rightarrow 1$ limit.

On might say that the set of orthogonal coordinate axis replaces the state space in quantum measurement. We do this replacement of space with coordinate axis all the time when at blackboard. Quantum consciousness theorist inside me adds that this means a creation of symbolic representations and that the function of quantum classical correspondences is to build symbolic representations for quantum reality at space-time level.

$q \rightarrow 1$ limit should have space-time correlates by quantum classical correspondence. A TGD inspired geometro-topological interpretation for the projection postulate might be that quantum measurement at $q \rightarrow 1$ limit corresponds to a leakage of 3-surface to a dark sector of imbedding space with $q \rightarrow 1$ (Planck constant near to 0 or ∞ depending on whether one has $n \rightarrow \infty$ covering or division of M^4 or CP_2 by a subgroup of $SU(2)$ becoming infinite cyclic - very roughly!) and Hilbert space is indeed effectively replaced with n rays. For $q \neq 1$ one would have only probabilities for different outcomes since things would be fuzzy.

In this picture classical physics and classical logic would be the physical counterpart for the shadow world of mathematics and would result only as an asymptotic notion.

10.2.3 Could 1-element fields actually correspond to Galois fields associated with infinite primes?

Finite field G_p corresponds to integers modulo p and product and sum are taken only modulo p. An alternative representation is in terms of phases $exp(ik2\pi/p)$, $k = 0, ..., p-1$ with sum and product performed in the exponent. The question is whether could one define these fields also for infinite primes by identifying the elements of this field as phases $exp(ik2\pi/\Pi)$ with k taken to be finite integer and Π an infinite prime (recall that they form infinite hierarchy). Formally this makes sense. 1-element field would be replaced with infinite hierarchy of Galois fields with infinite number of elements!

The probabilities defined by components of quantum spinor make sense only as real numbers and one can indeed map them to real numbers by interpreting q as an ordinary complex number. This would give same results as $q \rightarrow 1$ limit and one would have effectively 1-element field but actually a Galois field with infinite number of elements.

If one allows k to be also infinite integer but not larger than than Π in the real sense, the phases $exp(ik2\pi/\Pi)$ would be well defined as real numbers and could differ from 1. All real numbers in the range $[-1, 1]$ would be obtained as values of $cos(k2\pi/\Pi)$ so that this limit would effectively give real numbers.

This relates also interestingly to the question whether the notion of p-adic field makes sense for infinite primes. The p-adic norm of any infinite-p p-adic number would be power of π either infinite, zero, or 1. Excluding infinite normed numbers one would have effectively only p-adic integers in the range $1, ...\Pi - 1$ and thus only the Galois field G_Π representable also as quantum phases.

I conclude with a nice string of text from John'z page:

What's a mathematical phantom? According to Wraith, it's an object that doesn't exist within a given mathematical framework, but nonetheless "obtrudes its effects so convincingly that one is forced to concede a broader notion of existence".

and unashamedly propose that perhaps Galois fields associated with infinite primes might provide this broader notion of existence! In equally unashamed tone I ask whether there exists also hierarchy of conscious entities at $q = 1$ levels in real sense and whether we might identify ourselves as this kind of entities? Note that if cognition corresponds to p-adic space-time sheets, our cognitive bodies have literally infinite geometric size in real sense.

10.2.4 One-element field realized in terms of real units with number theoretic anatomy

One-element field looks rather self-contradictory notion since 1 and 0 should be represented by same element. The real units expressible as ratios of infinite rationals could however provide a well-defined realization of this notion.

1. The condition that same element represents the neutral element of both sum and product gives strong constraint on one-element field. Consider an algebra formed by reals with sum and product defined in the following manner. Sum, call it $\oplus$, corresponds to the ordinary product $x \times y$ for reals whereas product, call it $\otimes$, is identified as the non-commutative product $x \otimes y = x^y$. $x = 1$ represents both the neutral element (0) of $\oplus$ and the unit of $\otimes$. The sub-algebras generated by 1 and multiple powers $P_n(x) = P_{n-1}(x) \otimes x = x \otimes ... \otimes x$ form commutative sub-algebras of this algebra. When one restricts the consideration to $x = 1$ one obtains one-element field as sub-field which is however trivial since $\oplus$ and $\otimes$ are identical operations in this subset.

2. One can get over this difficulty by keeping the operations $\oplus$ and $\otimes$, by assuming one-element property only with respect to the real and various p-adic norms, and by replacing ordinary real unit 1 with the algebra of real units formed from infinite primes by requiring that the real and various p-adic norms of the resulting numbers are equal to one. As far as real and various p-adic norms are considered, one has commutative one-element field. When number theoretic anatomy is taken into account, the algebra contains infinite number of elements and is non-commutative with respect to the product since the number theoretic anatomies of x^y and y^x are different.

10.3 A Little Crazy Speculation About Knots And Infinite Primes

D-dimensional knots correspond to the isotopy equivalence classes of the imbeddings of spheres S^d to S^{d+2}. One can consider also the isotopy equivalence classes of more general manifolds $M^d \subset M^{d+2}$. Knots [A3] are very algebraic objects. The product (or sum, I prefer to talk about product) of knots is defined in terms of connected sum. Connected sum quite generally defines a commutative and associative product, and one can decompose any knot into prime knots.

Knots can be mapped to Jones polynomials $J(K)$ (for instance - there are many other polynomials and there are very general mathematical results about them [A3]) and the product of knots is mapped to a product of corresponding polynomials. The polynomials assignable to prime knots should be prime in a well-defined sense, and one can indeed define the notion of primeness for polynomials $J(K)$: prime polynomial does not factor to a product of polynomials of lower degree in the extension of rationals considered.

This raises the idea that one could define the notion of zeta function for knots. It would be simply the product of factors $1/(1-J(K)^{-s})$ where K runs over prime knots. The new (to me) but very natural element in the definition would be that ordinary prime is replaced with a polynomial prime. This observation led to the idea that the hierarchy of infinite primes could correspond to the hierarchy of knots in various dimensions and this in turn stimulated quite fascinating speculations.

10.3.1 Do knots correspond to the hierarchy of infinite primes?

A very natural question is whether one could define the counterpart of zeta function for infinite primes. The idea of replacing primes with prime polynomials would resolve the problem since infinite primes can be mapped to polynomials. For some reason this idea however had not occurred to me earlier.

The correspondence of both knots and infinite primes with polynomials inspires the question whether $d = 1$-dimensional prime knots might be in correspondence (not necessarily 1-1) with infinite primes. Rational or Gaussian rational infinite primes would be naturally selected these are also selected by physical considerations as representatives of physical states although quaternionic and octonionic variants of infinite primes can be considered.

If so, knots could correspond to the subset of states of a super-symmetric arithmetic quantum field theory with bosonic single particle states and fermionic states labeled by quaternionic primes.

1. The free Fock states of this QFT are mapped to first order polynomials and irreducible polynomials of higher degree have interpretation as bound states so that the non-decomposability to a product in a given extension of rationals would correspond physically to the non-decomposability into many-particle state. What is fascinating that apparently free arithmetic QFT allows huge number of bound states.

2. Infinite primes form an infinite hierarchy, which corresponds to an infinite hierarchy of second quantizations for infinite primes meaning that n-particle states of the previous level define single particle states of the next level. At space-time level this hierarchy corresponds to a hierarchy defined by space-time sheets of the topological condensate: space-time sheet containing a galaxy can behave like an elementary particle at the next level of hierarchy.

3. Could this hierarchy have some counterpart for knots?In one realization as polynomials, the polynomials corresponding to infinite prime hierarchy have increasing number of variables. Hence the first thing that comes into my uneducated mind is as the hierarchy defined by the increasing dimension d of knot. All knots of dimension d would in some sense serve as building bricks for prime knots of dimension $d + 1$ or possibly $d + 2$ (the latter option turns out to be the more plausible one). A canonical construction recipe for knots of higher dimensions should exist.

4. One could also wonder whether the replacement of spherical topologies for d-dimensional knot and $d + 2$-dimensional imbedding space with more general topologies could correspond to algebraic extensions at various levels of the hierarchy bringing into the game more general infinite primes. The units of these extensions would correspond to knots which involve in an essential manner the global topology (say knotted non-contractible circles in 3-torus). Since the knots defining the product would in general have topology different from spherical topology the product of knots should be replaced with its category theoretical generalization making higher-dimensional knots a groupoid in which spherical knots would act diagonally leaving the topology of knot invariant. The assignment of d-knots with the notion of n-category, n-groupoid, etc.. by putting d=n is a highly suggestive idea. This is indeed natural since are an outcome of a repeated abstraction process: statements about statements about.....

5. The lowest ($d = 1, D = 3$) level would be the fundamental one and the rest would be (somewhat boring!) repeated second quantization. This is why the dimension $D = 3$ (number theoretic braids at light-like 3-surfaces!) would be fundamental for physics.

10.3.2 Further speculations

Some further speculations about the proposed structure of all structures are natural.

1. The possibility that algebraic extensions of infinite primes could allow to describe the refinements related to the varying topologies of knot and imbedding space would mean a deep connection between number theory, manifold topology, sub-manifold topology, and n-category theory.

2. Category theory appears already now in fundamental role in the construction of the generalization of M-matrix unifying the notions of density matrix and S-matrix. Generalization of category to n-category theory and various n-structures would have very direct correspondence with the physics of TGD Universe if one assumes that repeated second quantization makes sense and corresponds to the hierarchical structure of many-sheeted space-time where even galaxy corresponds to elementary fermion or boson at some level of hierarchy.

   This however requires that the unions of light-like 3-surfaces and of their sub-manifolds at different levels of topological condensate are able to represent higher-dimensional manifolds physically albeit not in the standard geometric sense since imbedding space dimension is just 8. This might be possible.

3. As far as physics is considered, the disjoint union of sub-manifolds of dimensions d_1 and d_2 behaves like a $d_1 + d_2$-dimensional Cartesian product of the corresponding manifolds. This is of course used in standard manner in wave mechanics (the WCW of n-particle system is identified as E^{3n}/S_n with division coming from statistics).

4. If the surfaces have intersection points, one has a union of Cartesian product with punctures (intersection points) and of lower-dimensional manifold corresponding to the intersection points.

5. Note also that by posing symmetries on classical fields one can effectively obtain from a given n-manifold manifolds (and orbifolds) with quotient topologies.

The megalomanic conjecture is that this kind of physical representation of d-knots and their imbedding spaces is possible using many-sheeted space-time. Perhaps even the entire magnificent mathematics of n-manifolds and their sub-manifolds might have a physical representation in terms of sub-manifolds of 8-D $M^4 \times CP_2$ with dimension not higher than space-time dimension $d = 4$.

10.3.3 The idea survives the most obvious killer test

All this looks nice and the question is how to give a death blow to all this reckless speculation. Torus knots are an excellent candidate for performing this unpleasant task but the hypothesis survives!

1. Torus knots [A7] are labeled by a pair integers (m, n), which are relatively prime. These are prime knots. Torus knots for which one has $m/n = r/s$ are isotopic so that any torus knot is isotopic with a knot for which m and n have no common prime power factors.

2. The simplest infinite primes correspond to free Fock states of the supersymmetric arithmetic QFT and are labeled by pairs (m, n) of integers such that m and n do not have any common prime factors. Thus torus knots would correspond to free Fock states! Note that the prime power $p^{k(p)}$ appearing in m corresponds to $k(p)$-boson state with boson "momentum" p and the corresponding power in n corresponds to fermion state plus $k(p) - 1$ bosons.

3. A further property of torus knots is that (m, n) and (n, m) are isotopic: this would correspond at the level of infinite primes to the symmetry $mX + n \to nX + m$, X product of all finite primes. Thus infinite primes are in $2 \to 1$ correspondence with torus knots and the hypothesis survives also this murder attempt. Probably the assignment of orientation to the knot makes the correspondence 1-1 correspondence.

10.3.4 How to realize the representation of the braid hierarchy in many-sheeted space-time?

One can consider a concrete construction of higher-dimensional knots and braids in terms of the many-sheeted space-time concept.

1. The basic observation is that ordinary knots can be constructed as closed braids so that everything reduces to the construction of braids. In particular, any torus knot labeled by (m, n) can be made from a braid with m strands: the braid word in question is $(\sigma_1....\sigma_{m-1})^n$ or by $(m,n)=(n,m)$ equivalence from n strands. The construction of infinite primes suggests that also the notion of d-braid makes sense as a collection of d-braids in $d+2$-space, which move and and define $d+1$-braid in $d+3$ space (the additional dimension being defined by time coordinate).

2. The notion of topological condensate should allow a concrete construction of the pairs of d- and $d+2$-dimensional manifolds. The 2-D character of the fundamental objects (partons) might indeed make this possible. Also the notion of length scale cutoff fundamental for the notion of topological condensate is a crucial element of the proposed construction.

3. Infinite primes have also interpretation as physical states and the representation in terms of knots would mean a realization of quantum classical correspondence.

The concrete construction would proceed as follows.

1. Consider first the lowest non-trivial level in the hierarchy. One has a collection of 3-D light-like 3-surfaces X_i^3 representing ordinary braids. The challenge is to assign to them a 5-D imbedding space in a natural manner. Where do the additional two dimensions come from? The obvious answer is that the new dimensions correspond to the partonic 2-surface X^2 assignable to the $3-D$ light-like surface X^3 at which these surfaces have suffered topological condensation. The geometric picture is that X_i^3 grow like plants from ground defined by X^2 at 7-dimensional $\delta M_+^4 \times CP_2$.

2. The degrees of freedom of X^2 should be combined with the degrees of freedom of X_i^3 to form a 5-dimensional space X^5. The natural idea is that one first forms the Cartesian products $X_i^5 = X_i^3 \times X^2$ and then the desired 5-manifold X^5 as their union by posing suitable additional conditions. Braiding means a translational motion of X_i^3 inside X^2 defining braid as the orbit in X^5. It can happen that X_i^3 and X_j^3 intersect in this process. At these points of the union one must obviously pose some additional conditions. Same applies to intersection of more than two X_i^3.

   Finite (p-adic) length scale resolution suggests that all points of the union at which an intersection between two or more light-like 3-surfaces occurs must be regarded as identical. In general the intersections would occur in a 2-d region of X^2 so that the gluing would take place along 5-D regions of X_i^5 and there are therefore good hopes that the resulting 5-D space is indeed a manifold. The imbedding of the surfaces X_i^3 to X^5 would define the braiding.

3. At the next level one would consider the 5-d structures obtained in this manner and allow them to topologically condense at larger 2-D partonic surfaces in the similar manner. The outcome would be a hierarchy consisting of $2n+1$-knots in $2n+3$ spaces. A similar construction applied to partonic surfaces gives a hierarchy of $2n$-knots in $2n+2$-spaces.

4. The notion of length scale cutoff is an essential element of the many-sheeted space-time concept. In the recent context it suggests that d-knots represented as space-time sheets topologically condensed at the larger space-time sheet representing $d+2$-dimensional imbedding space could be also regarded effectively point-like objects (0-knots) and that their d-knottiness and internal topology could be characterized in terms of additional quantum numbers. If so then d-knots could be also regarded as ordinary colored braids and the construction at higher levels would indeed be very much analogous to that for infinite primes.

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