TGD leads to several proposals for the exact solution of field equations defining space-time surfaces as preferred extremals of twistor lift of K\"ahler action. So called M^8-H duality is one of these approaches. The beauty of M^8-H duality is that it could reduce classical TGD to algebraic geometry and would immediately provide deep insights to cognitive representation identified as sets of rational points of these surfaces.

In the sequel I shall consider the following topics.

\begin{enumerate}

\item I will discuss basic notions of algebraic geometry such as algebraic variety, surface, and curve, all rational point of variety central for TGD view about cognitive representation, elliptic curves and surfaces, and rational and potentially rational varieties. Also the notion of Zariski topology and Kodaira dimension are discussed briefly. I am not a mathematician and what hopefully saves me from horrible blunders is physical intuition developed during 4 decades of TGD.

\item It will be shown how M^8-H duality could reduce TGD at fundamental level to octonionic algebraic geometry. Space-time surfaces in M^8 would be algebraic surfaces identified as zero loci for imaginary part $IM(P)$ or real part $RE(P)$ of octonionic polynomial of complexified octonionic variable o_c decomposing as $o_c= q^1_c+q^2_c I^4$ and projected to a Minkowskian sub-space M^8 of complexified O. Single real valued polynomial of real variable with algebraic coefficients would determine space-time surface! As proposed already earlier, spacetime surfaces would form commutative and associative algebra with addition, product and functional composition.

One can interpret the products of polynomials as correlates for free many-particle states with interactions described by added interaction polynomial, which can vanish at boundaries of CDs thanks to the vanishing in Minkowski signature of the complexified norm $q_c\overline{q_c}$ appearing in $RE(P)$ or $IM(P)$ caused by the quaternionic non-commutativity. This leads to the same picture as the view about preferred extremals reducing to minimal surfaces near boundaries of CD. Also zero zero energy ontology (ZEO) could emerge naturally from the failure of number field property for for quaternions at light-cone boundaries.

\end{enumerate}

The construction and interpretation of the octonionic geometry involves several challenges.

\begin{enumerate}

\item   The fundamental challenge is to prove that the octonionic polynomials with real coefficients determine associative (co-

associative) surfaces as the zero loci of their real part $RE(P)$ (imaginary parts $IM(P)$). $RE(P)$ and $IM(P)$ are defined in quaternionic sense. Contrary to the first naive working hypothesis, the identification $M^4 \subset O$ as as a co-associative region turns out to be the correct choice making light-cone boundary a counterpart of point-like singularity
essential for the emergence of causal diamonds (CDs).

The hierarchy of notions involved is well-ordering for 1-D structures, commutativity for complex numbers, and associativity for quaternions. This suggests a generalization of Cauchy-Riemann conditions for complex analytic functions to quaternions and octonions. Cauchy Riemann conditions are linear and constant value manifolds are 1-D and thus well-ordered. Quaternionic polynomials with real coefficients define maps for which the 2-D spaces corresponding to vanishing of real/imaginary parts of the polynomial are complex/co-complex or equivalently commutative/co-commutative. Commutativity is expressed by conditions bilinear in partial derivatives. Octonionic polynomials with real coefficients define maps for which 4-D surfaces for which real/imaginary part are quaternionic/co-quaternionic, or equivalently associative/co-associative. The conditions are now 3-linear.

In fact, all algebras obtained by Cayley-Dickson construction adding imaginary units to octonionic algebra are power associative so that polynomials with real coefficients define an associative and commutative algebra. Hence octonion analyticity and M^8-H correspondence could generalize.

\item It turns out that in the generic case associative surfaces are 3-D and are obtained by requiring that one of the coordinates $RE(Y)^i$ or $IM(Y)^i$ in the decomposition $Y^i=RE(Y)^i+IM(Y)^iI_4$ of the gradient of $RE(P)= Y=0$ with respect to the complex coordinates z_i^k, $k=1,2$, of O vanishes that is critical as function of quaternionic components z_1^k or z_2^k associated with q_1 and q_2 in the decomposition $o= q_1+q_2I_4$, call this component X_i. In the generic case this
gives 3-D surface.

In this generic case M^8-H duality can map only the 3-surfaces at the boundaries of CD and light-like partonic orbits to H, and only determines the boundary conditions of the dynamics in H determined by the twistor lift of K\"ahler action. M^8-H duality would allow to solve the gauge conditions for SSA (vanishing of infinite number of Noether charges) explicitly.

One can also have criticality. 4-dimensionality can be achieved by posing conditions on the coefficients of the octonionic polynomial P so that the criticality conditions do not reduce the dimension: X_i would have possibly degenerate zero at space-time variety. This can allow 4-D associativity with at most 3 critical components X_i. Space-time surface would be analogous to a polynomial with a multiple root. The criticality of X_i conforms with the general vision about quantum criticality of TGD Universe and provides polynomials with universal dynamics of criticality. A generalization of Thom's catastrophe theory emerges. Criticality should be equivalent to the universal dynamics determined by the twistor lift of K\"ahler action in H in regions, where K\"ahler action and volume term decouple and dynamics does not depend on coupling constants.

One obtains two types of space-time surfaces. Critical and associative (co-associative) surfaces can be mapped by M^8-H duality to preferred critical extremals for the twistor lift of K\"ahler action obeying universal dynamics with no dependence on coupling constants and due to the decoupling of K\"ahler action and volume term: these represent external particles. M^8-H duality does not apply to non-associative (non-co-associative) space-time surfaces except at 3-D boundary surfaces. These regions correspond to interaction regions in which K\"ahler action and volume term couple and coupling constants make themselves visible in the dynamics. M^8-H duality determines boundary conditions.

\item This picture generalizes to the level of complex/co-complex surfaces assigned with fermionic dynamics. Why in some cases 1-D light-like curves at partonic orbits seem to be enough to represent fermions? Why fermionic strings serve as correlates of entanglement for bound states? What selects string world sheets and partonic 2-surfaces from the slicing of space-time surfaces? I have proposed commutativity or co-commutatitivity of string worlds sheets/ partonic 2-surfaces in quaternionic sense as number theoretic explanation (tangent space as a sub-space of quaternionic space is commutative/co-commutative at each point). Why not all string world sheets/partonic 2-surfaces in the slicing are not commutative/co-commutative? The answer to these questions is criticality again: in the generic case commutative varieties are 1-D curves. In critical case one has 2-D string worlds sheets and partonic 2-surfaces.

\end{enumerate}

Also a sketchy proposal for the description of interactions is discussed.

\begin{enumerate}

\item $IM(P_1P_2)=0$ is satisfied for $IM(P_1)=0$ and $IM(P_2)=0$ since $IM(o_1o_2)$ is linear in $IM(o_i)$ and one obtains union of space-time varieties. $RE(P_1P_2)=0$ cannot be satisfied in this manner since $RE(o_1o_2)$ is not linear in $RE(o_i)$ so that the two varieties interact and this interaction could give rise to a wormhole contact connecting the two space-time varieties.

\item The surprise that $RE(P)=0$ and $IM(P)=0$ conditions have as singular solutions light-cone interior and its complement and 6-spheres $S^6(t_n)$ with radii t_n given by the roots of the real

$P(t)$, whose octonionic extension defines the space-time variety X^4. The intersections $X^2= X^4\cap S^6(t_n)$ is tentatively identified as partonic 2-varieties defining topological interaction vertices.

The idea about the reduction of zero energy states to discrete cognitive representations suggests that interaction vertices at partonic varieties X^2 are associated with the discrete set of intersection points of the sparticle lines at light-like orbits of partonic 2-surfaces belonging to extension of rationals.

\item CDs and therefore also ZEO emerge naturally. For CDs with different origins the products of polynomials fail to commute and associate unless the CDs have tips along real (time) axis. The first option is that all CDs under observation satisfy this condition. Second option allows general CDs.

The proposal is that the product $\prod P_i$ of polynomials associated with CDs with tips along real axis
the condition $IM(\prod P_i)=0$ reduces to $IM(P_i)=0$ and criticality conditions guaranteeing associativity and provides a description of the external particles. Inside these CDs $RE(\prod P_i)=0$ does not reduce to $RE(\prod P_i)=0$, which automatically gives rise to geometric interactions. For general CDs the situation is more complex.

\item The possibility of super-octonionic geometry raises the hope that the twistorial construction of scattering amplitudes in ${\cal N}=4$ SUSY generalizes to TGD in rather straightforward manner to a purely geometric construction. Functional integral over WCW would reduce to summations over polynomials with coefficients in extension of rationals and criticality conditions on the coefficients could make the summation well-defined by bringing in finite measurement resolution.

\end{enumerate}

Gromov-Witten (G-W) invariants belong to the realm of quantum enumerative geometry briefly discussed in \cite{btart}{ratpoints1}. They count numbers of points in the intersection of varieties (\blockquote{branes}) with quantum intersection identified as the existence of \blockquote{string world sheet(s)} intersecting the branes. Also octonionic geometry gives rise to brane like objects. G-W invariants are rational numbers but it is proposed that they could be integers in TGD framework.

Riemann-Roch theorem (RR) and its generalization Atyiah-Singer index theorem (AS) relate dimensions of various kinds of moduli spaces to topological invariants. The possible generalizations of RR and AS to octonionic framework and the implications of M^8-H duality for the possible generalizations are discussed. The adelic hierarchy of extensions of rationals and criticality conditions make the moduli spaces discrete so that one expects kind of particle in box type quantization selecting discrete points of moduli spaces about the dimension.